The Cosserat Eigenvalue Problem

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- François Cosserat 1852–1914
- Eugène Cosserat 1866–1931

MAURIZIO BROCATO, KONSTANTINOS CHATZIS.
Les frères Cosserat : brève introduction à leur vie et à leurs travaux en mécanique.
Introduction à la réédition de : E. et F. Cosserat, Théorie des corps déformables, Paris : Hermann, 2009 (1re éd., 1909), iii-xlv.

Eugène Cosserat

» Supposons, pour fixer les idées, qu'on se propose de trouver trois fonctions u, v, w remplissant les conditions de continuité fondamentales à l'égard du domaine constitué par un ellipsoïde à trois axes inégaux, prenant des valeurs données sur la frontière

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

de cet ellipsoïde, et satisfaisant aux équations

$$\Delta_2 u + \xi \frac{\partial \theta}{\partial x} = 0, \qquad \Delta_2 v + \xi \frac{\partial \theta}{\partial y} = 0, \qquad \Delta_2 w + \xi \frac{\partial \theta}{\partial z} = 0.$$

» Au point de vue où nous nous sommes placés, la principale difficulté du problème consiste dans la détermination *effective* d'une série de nombres k_i, tous différents de -1, et à chacun desquels on peut associer au moins un système de trois fonctions U_i, V_i, W_i s'annulant à la frontière et vérifiant les équations

$$(1) \quad \Delta_2 U_i + k_i \frac{\partial \Theta_i}{\partial x} = 0, \qquad \Delta_2 V_i + k_i \frac{\partial \Theta_i}{\partial y} = 0, \qquad \Delta_2 W_i + k_i \frac{\partial \Theta_i}{\partial z} = 0.$$

Original Formulation (E.&F. Cosserat 1898)

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$\theta = \operatorname{div} \boldsymbol{u}$

$\Delta \boldsymbol{u} + \xi \nabla \operatorname{div} \boldsymbol{u} = 0$

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7. Domains with $\sigma(\Omega) > 0$
   - Unions of domains
   - Bogovskiĭ's integral operator
8. Non-Smooth Domains
   - Corners and Essential Spectrum
   - The Horgan–Payne Angle
9. Majorants
   - Small Cuts
   - Cusps
   - Thin Domains
   - Rectangles
10. John Domains
    - Definition
    - Pictures
    - A Theorem

Part I

Historical Introduction

1. The Cosserat Eigenvalue Problem
   - Cosserat 1898
   - Modern formulations
   - Original Motivation: Eigenfunction Expansion

2. Historical Timeframe

3. Related Problems: Some Inequalities
   - Korn Inequality
   - Friedrichs Inequality
   - Babuška-Aziz–LBB inequality
   - Schur Complement for Stokes System, Uzawa

Equivalent formulations

1. Lamé equations: $\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\,\mathrm{div}\,\boldsymbol{u} = 0$
2. Cosserat: $\Delta\boldsymbol{u} + \xi\nabla\,\mathrm{div}\,\boldsymbol{u} = 0, \qquad \xi = (\lambda+\mu)/\mu$
3. Spectral problem: $\sigma\Delta\boldsymbol{u} - \nabla\,\mathrm{div}\,\boldsymbol{u} = 0, \qquad \sigma = -1/\xi$
4. Variational formulation: $\sigma\int\nabla\boldsymbol{u}:\nabla\boldsymbol{v} = \int\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} \quad \forall\boldsymbol{v}$

Find $\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{u}\not\equiv 0$, and $\sigma\in\mathbb{C}$ such that

$$\boxed{\sigma\Delta\boldsymbol{u} = \nabla\,\mathrm{div}\,\boldsymbol{u}} \quad \text{in } \Omega\subset\mathbb{R}^d$$

Easy to see: $0\le\sigma\le 1$. Obvious special values:

$\sigma = 0$: ∇ div has an infinite-dimensional kernel containing

$$\boldsymbol{H}_0^1(\mathrm{div}\,0,\Omega)\supset\mathbf{curl}\big(C_0^\infty(\Omega)^d\big) \quad (d=3)$$

$\sigma = 1$: $\Delta\boldsymbol{u} - \nabla\,\mathrm{div} = -\mathbf{curl}\,\mathbf{curl}$ has an infinite-dimensional kernel containing

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Find $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$, $\boldsymbol{u} \not\equiv 0$, and $\sigma \in \mathbb{C}$ such that

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Guiding example: Laplace equation

The problem

$$u \in H_0^1(\Omega): \quad \Delta u + \kappa u = f$$

has the solution

$$u(x) = \sum_{j=1}^{\infty} \frac{f_j}{\kappa - \lambda_j} u_j(x)$$

if $f(x) = \sum f_j u_j(x)$ is the expansion of f in eigenfunctions u_j of $-\Delta$ with eigenvalues λ_j.

E.&F. Cosserat derive a similar expansion for a solution of the Lamé equations with given data $\boldsymbol{u}_0$ on the boundary:

$$(2) \qquad u = u_0 + \xi \sum_{i=1}^{i=\infty} \frac{k_i \mathrm{U}_i}{\xi - k_i}$$

Eigenfunction Expansion for a Ball

Lemma

On the ball $\Omega = B_R(0) \subset \mathbb{R}^d$, if p is a harmonic polynomial homogeneous of degree k, the solution of the Dirichlet problem

$$\Delta u = p \;\; in\, \Omega, \quad u = 0 \;\; on\, \partial\Omega$$

is given by

$$u(x) = c\,(|x|^2 - R^2)\,p(x)\,, \quad c = \frac{1}{2d + 4k}\,.$$

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Proof : $\Delta u = c\,(\Delta|x|^2 p + 2\nabla|x|^2 \cdot \nabla p)$
$= c\,(\quad 2d\,p + \qquad\quad 4k\,p)$
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$\Delta : (|x|^2 - R^2)\dot{\mathbb{P}}_k \to \dot{\mathbb{P}}_k$ is injective, hence bijective.

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Let p be a harmonic polynomial homogeneous of degree k and

$$\boldsymbol{v}(x) = (|x|^2 - R^2)\,\nabla p(x)$$

Then v satisfies

$$\sigma_k \Delta \boldsymbol{v} = \nabla \operatorname{div} \boldsymbol{v} \quad with\ \sigma_k = \frac{k}{d + 2k - 2}$$

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Proof : We have seen

$$\Delta v = (2d + 4(k-1))\,\nabla p$$

We compute

$$\operatorname{div} v = \nabla|x|^2 \cdot \nabla p + (|x|^2 - R^2)\Delta p = 2k\,p$$

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$$\operatorname{div} \Delta^{-1} \nabla p = \sigma_k p$$

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Remark (to be remembered...)

The scalar harmonic function p satisfies

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Corollary

Let $\boldsymbol{u}_0 \in L^2(\partial B_R(0))$ and write $\mathrm{H}\boldsymbol{u}_0$ for its harmonic extension to $B_R(0)$. Define $p_0 = \operatorname{div} \mathrm{H}\boldsymbol{u}_0$ and let

$$p_0(x) = \sum_{k \geq 1} p_k(x)$$

be its expansion in harmonic polynomials (spherical harmonics!). Let $\boldsymbol{v}_k = (|x|^2 - R^2)\nabla p_k$. Then for $\sigma \notin \{\sigma_k\}$, the function

$$\boldsymbol{u}(x) = \mathrm{H}\boldsymbol{u}_0(x) - \sum_{k \geq 1} \frac{\sigma_k}{2k(\sigma_k - \sigma)} \boldsymbol{v}_k(x).$$

solves

$$\sigma \Delta \boldsymbol{u} - \nabla \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } B_R(0), \qquad \boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on } \partial B_R(0)$$

Observation

For all d: $\sigma_1 = \frac{1}{d} \leq \sigma_k \to \frac{1}{2}$. For $d = 2$, all σ_k are equal to $\frac{1}{2}$.

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Eigenfunction Expansion on a Smooth Domain

Theorem (Mikhlin 1973)

Let Ω be a smooth bounded domain.
Then the Cosserat eigenvalue problem has a sequence of eigenfunctions forming an orthonormal basis of $\boldsymbol{L}^2(\Omega)$ and also an orthogonal basis of $\boldsymbol{H}_0^1(\Omega)$.
The Cosserat eigenvalues satisfy $\sigma \in [0,1]$.
The values $\sigma = 0$ and $\sigma = 1$ are isolated eigenvalues of infinite multiplicity, and there is a sequence of eigenvalues converging to $\sigma = \frac{1}{2}$.

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Time Frame: Milestones

1898-1901	E.&F. Cosserat: 9 papers in CR Acad Sci Paris
1924	L. Lichtenstein: a boundary integral equation method
1967	V. Maz'ya – S. Mikhlin: "On the Cosserat spectrum. . . "
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† It is the analogue of the inequality of A. Korn for functions of three variables. The expansion theorem is related to those of E. and F. Cosserat.

Cf. A. Korn, *Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen*, Bulletin de l'Académie des Sciences de Cracovie, 1909, vol. 2, pp. 705–724, and literature indicated therein.

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1994-2000	E. Chizhonkov – V. Ol'shanskiĭ: "On the optimal constant in the inf-
1999-2009	G. Stoyan: discrete inequalities
2000-2004-	S. Zsuppán: conformal mappings
2006-	C. Simader – W. v. Wahl – S. Weyers: L^q, unbounded domains
2006-	G. Acosta – R.G. Durán – M.A. Muschietti: John domains
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- We denote by $\boldsymbol{e}(\boldsymbol{u})$ the linearized strain tensor of $\boldsymbol{u}$

$$e_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i), \quad 1 \le i,j \le d,$$

- We denote by $\boldsymbol{r}(\boldsymbol{u})$ its antisymmetric counterpart (related to $\mathbf{curl}\,\boldsymbol{u}$)

$$r_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(\partial_i u_j - \partial_j u_i), \quad 1 \le i,j \le d,$$

Definition

Let Ω be a domain in $\mathbb{R}^d$. It is said to satisfy the second Korn inequality if there exists a positive constant K such that for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisfying the condition

$$\int_\Omega r_{ij}(\boldsymbol{u})(x)\,dx = 0, \quad 1 \le i,j \le d$$

there holds the estimate

$$\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} \le K \|\boldsymbol{e}(\boldsymbol{u})\|^2_{L^2(\Omega)}$$

If such a K exists we denote by $K(\Omega)$ the smallest such K.

Korn Inequality

Theorem (Korn – Friedrichs – Nečas – Nitsche)

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then

$$K(\Omega) < \infty .$$

Corollary

If the Lamé constants λ, μ are positive, then the Neumann problem for the Lamé equations

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu)\nabla \operatorname{div} \boldsymbol{u} = f \ \text{ in } \Omega; \quad \text{normal stress zero on } \partial\Omega$$

has a strongly elliptic variational formulation in $\boldsymbol{H}^1(\Omega)$. It is well-posed in any closed subspace of $\boldsymbol{H}^1(\Omega)$ that does not contain rigid motions. Consequences : Fredholm alternative, discrete eigenfrequencies in elastodynamics, convergence of finite element approximations, ...

The energy quadratic form is

$$2\mu \|\boldsymbol{e}(\boldsymbol{u})\|^2_{L^2(\Omega)} + \lambda \|\operatorname{div} \boldsymbol{u}\|^2_{L^2(\Omega)} \ge \frac{2\mu}{K(\Omega)} \|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} = \frac{2\mu}{K(\Omega)} \big(\|\boldsymbol{u}\|^2_{H^1(\Omega)} - \|\boldsymbol{u}\|^2_{L^2(\Omega)}\big)$$

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Let $\Omega \subset \mathbb{R}^2$. Consider holomorphic functions w with real part f and imaginary part g:

$$w(z) = f(z) + i\,g(z)$$

Definition

Let Ω be a domain in $\mathbb{R}^2$. It is said to satisfy the Friedrichs inequality if there exists a positive constant Γ such that for all holomorphic $w \in L^2(\Omega)$ satisfying the condition

$$\int_\Omega f(x)\,dx = 0$$

there holds the estimate

$$\|f\|^2_{L^2(\Omega)} \le \Gamma \|g\|^2_{L^2(\Omega)}$$

If such a Γ exists we denote by $\Gamma(\Omega)$ the smallest such Γ.

The Friedrichs inequality holds for any bounded Lipschitz domain in $\mathbb{R}^2$.

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Babuška-Aziz–LBB inequality

Define

$$L^2_\circ(\Omega) = \{u \in L^2(\Omega) \mid \int_\Omega u(x)\,dx = 0\}$$

Definition

Let Ω be a domain in $\mathbb{R}^d$. It is said to satisfy the Babuška-Aziz inequality if there exists a positive constant β such that for all $q \in L^2_\circ(\Omega) \setminus \{0\}$ there exists a $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \setminus \{0\}$ with

$$\beta \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)} \le \int_\Omega (\operatorname{div} \boldsymbol{v})(x)\, q(x)\, dx\,.$$

We denote by $\beta(\Omega)$ the largest such β:

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v})(x)\, q(x)\, dx}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}$$

$\beta(\Omega)$ is the LBB constant or inf-sup constant of Ω.

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We denote by $\beta(\Omega)$ the largest such β:

$$\beta(\Omega)^{-1} = \min\{\|B\| \mid B : L^2_\circ(\Omega) \to \boldsymbol{H}^1_0(\Omega) \text{ is a right inverse of the div operator}\}$$

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For any bounded Lipschitz domain in $\mathbb{R}^d$ there holds $0 < \beta(\Omega) < \infty$.

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Theorem (Babuška-Aziz – Payne-Weinberger)

For any bounded Lipschitz domain in $\mathbb{R}^d$ there holds $0 < \beta(\Omega) < \infty$.

Proof of equivalence : Implications for $\beta > 0$:

$$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6 \Rightarrow 1$$

1. $$\inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v})(x)\, q(x)\, dx}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta$$
2. $$\forall q \in L^2_\circ(\Omega): \quad \sup_{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v})(x)\, q(x)\, dx}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} \geq \beta \|q\|_{L^2(\Omega)}$$
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Consider the Stokes problem for $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$, $p \in L^2_\circ(\Omega)$:

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla p = \mathbf{f} \qquad & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \qquad & \text{in } \Omega \end{aligned}$$

Pressure Stability for Stokes problem

Let $\nu > 0$ and let Ω be such that $\beta(\Omega) > 0$. Let C_P be the constant in the Poincaré inequality

$$\|v\|_{L^2(\Omega)} \le C_P \, \|\nabla v\|_{L^2(\Omega)} \qquad \forall\, v \in H_0^1(\Omega)\,.$$

Then for $f \in \boldsymbol{L}^2(\Omega)$ there exists a unique solution $(\boldsymbol{u}, p)$ of the Stokes problem, and

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \le \frac{C_P}{\nu}\, \|f\|_{L^2(\Omega)}$$

$$\|p\|_{L^2(\Omega)} \le \frac{2C_p}{\beta(\Omega)}\, \|f\|_{L^2(\Omega)}$$

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Proof of the estimates : Write $|u|$ for the $L^2(\Omega)$-norm of u and $|u|_1 = |\nabla u|$ for its $H^1(\Omega)$-seminorm. Variational form of Stokes:

$$\forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \quad \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - \int_\Omega p \operatorname{div} \boldsymbol{v} = \int_\Omega \mathbf{f} \cdot \boldsymbol{v}$$

Taking $\boldsymbol{v} = \boldsymbol{u}$, one gets

$$\nu |\boldsymbol{u}|_1^2 = \int_\Omega \mathbf{f} \cdot \boldsymbol{u} \le |\mathbf{f}||\boldsymbol{u}| \le |\mathbf{f}| C_P |\boldsymbol{u}|_1$$

and there exists $\boldsymbol{v}$ such that

$$\beta(\Omega)|p||\boldsymbol{v}|_1 \le \int_\Omega p \operatorname{div} \boldsymbol{v} = \int_\Omega \mathbf{f} \cdot \boldsymbol{v} - \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \le |\mathbf{f}||\boldsymbol{v}| + \nu |\boldsymbol{u}|_1 |\boldsymbol{v}|_1 \le 2C_P |\mathbf{f}||\boldsymbol{v}|_1$$

The Uzawa algorithm for solving the Stokes problem is the iteration

$$
\begin{aligned}
-\nu\Delta \boldsymbol{u}_{n+1} &= \mathbf{f} - \nabla p_n \\
p_{n+1} &= p_n - \rho_n \nu \operatorname{div} \boldsymbol{u}_{n+1}
\end{aligned}
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Here $\rho_n > 0$ are suitably chosen relaxation parameters.

Definition

The Schur complement operator $\mathcal{S}$ for the Stokes system is

$$\mathcal{S} = \operatorname{div}\Delta^{-1}\nabla : \quad L^2_\circ \xrightarrow{\nabla} H^{-1} \xrightarrow{\Delta^{-1}} H^1_0 \xrightarrow{\operatorname{div}} L^2_\circ$$

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This means that $\mathscr{S} q = \operatorname{div} \boldsymbol{w}$, where $\boldsymbol{w} \in \boldsymbol{H}^1_0$ is the solution of the Dirichlet problem $\Delta \boldsymbol{w} = \nabla q$, or in variational form

$$\forall\, \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) : \quad \int_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v} = \int_\Omega q \operatorname{div} \boldsymbol{v}\,.$$

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From Stokes and Uzawa one gets

$$0 = \nu \operatorname{div} \boldsymbol{u} = \operatorname{div}\Delta^{-1}(\nabla p - \mathbf{f}) = \mathscr{S}p - \operatorname{div}\Delta^{-1}\mathbf{f}$$
$$p_{n+1} = p_n - \rho_n(\mathscr{S}p_n - \operatorname{div}\Delta^{-1}\mathbf{f}) = p_n - t\rho_n\mathscr{S}(p_n - p)$$
$$\Longrightarrow \quad p - p_{n+1} = \big(I - \rho_n\mathscr{S}\big)(p - p_n)$$

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$I - \rho_n \mathscr{S}$ is the error reduction operator of the Uzawa algorithm

$$\big|p - p_{n+1}\big| \le \max_{\sigma \in \mathrm{Sp}(\mathscr{S})} \big|1 - \rho_n \sigma\big| \, \big|p - p_n\big|$$

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Here $\rho_n > 0$ are suitably chosen relaxation parameters.

Definition

The Schur complement operator $\mathscr{S}$ for the Stokes system is

$$\mathscr{S} = \operatorname{div}\Delta^{-1}\nabla : \quad L^2_\circ \xrightarrow{\nabla} \boldsymbol{H}^{-1} \xrightarrow{\Delta^{-1}} \boldsymbol{H}^1_0 \xrightarrow{\operatorname{div}} L^2_\circ$$

Conclusion

Error analysis of the Uzawa algorithm

$\Longleftrightarrow$

Analysis of the spectrum $\mathrm{Sp}(\mathscr{S})$ of the Schur complement

Part II

Cosserat Spectrum and Related Problems

Let $\boldsymbol{u}$ satisfy the Lamé equations in the Cosserats notation

$$\Delta \boldsymbol{u} + \xi \nabla \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on } \partial\Omega$$

Observation : If $\xi \neq -1$ then $\theta = \operatorname{div} \boldsymbol{u}$ satisfies $\Delta\theta = 0$.

$$\theta(x) = \mathrm{H}\theta_0(x) = \int_{\partial\Omega} \partial_{n(y)} G(x,y)\, \theta_0(y)\, ds(y) \qquad (x \in \Omega)$$

where H means harmonic extension, $G(x,y)$ denotes the Green function for the Dirichlet problem in Ω, and $\theta_0 = \gamma\theta = \theta\big|_{\partial\Omega}$.

Trick : Define $\boldsymbol{w} = \boldsymbol{u} + \kappa \boldsymbol{x} \theta$. $\Rightarrow \Delta \boldsymbol{w} = \Delta \boldsymbol{u} + 2\kappa \nabla\theta = 0$ if $\kappa = \xi/2$.

$$\boldsymbol{w}(x) = \mathrm{H}(\boldsymbol{u}_0 + \kappa \boldsymbol{x} \theta_0)(x) = \mathrm{H}\boldsymbol{u}_0(x) + \int_{\partial\Omega} \partial_{n(y)} G(x,y)\, \kappa \boldsymbol{y} \theta_0(y)\, ds(y) \qquad (x \in \Omega)$$

Also $\operatorname{div} \boldsymbol{w} = \operatorname{div} \boldsymbol{u} + \kappa d\theta + \kappa \boldsymbol{x} \cdot \nabla\theta$

$= (1 + d\kappa)\theta + \kappa \int_{\partial\Omega} \boldsymbol{x} \cdot \nabla_x \partial_{n(y)} G(x,y)\, \theta_0(y)\, ds(y).$

On the other hand

$\operatorname{div} \boldsymbol{w} = \operatorname{div} \mathrm{H}\boldsymbol{u}_0 + \kappa \int_{\partial\Omega} \boldsymbol{y} \cdot \nabla_x \partial_{n(y)} G(x,y)\, \theta_0(y)\, ds(y)$

Lichtenstein's integral equation [MZ 20, 1924]

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$$\operatorname{div}\boldsymbol{w}=\operatorname{div}\mathrm{H}\boldsymbol{u}_0+\kappa\int_{\partial\Omega}\boldsymbol{y}\cdot\nabla_x\partial_{n(y)}G(x,y)\,\theta_0(y)\,ds(y)$$

$$(1 + d\kappa)\theta(x) + \kappa \int_{\partial\Omega} L(x,y)\,\theta_0(y)\,ds(y) = \operatorname{div} \mathsf{H}\boldsymbol{u}_0(x) \qquad (x \in \Omega)$$

with the kernel

$$L(x,y) = (\boldsymbol{x} - \boldsymbol{y}) \cdot \nabla_x \partial_{n(y)} G(x,y)$$

Singularity: $L(x,y) \sim (1-d)\partial_{n(y)} G(x,y)$.

Lichtenstein's integral equation [MZ 20, 1924]

$$(1+d\kappa)\theta(x)+\kappa\int_{\partial\Omega} L(x,y)\,\theta_0(y)\,ds(y)=\operatorname{div}\mathsf{H}\boldsymbol{u}_0(x)\qquad(x\in\Omega)$$

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Trace on the boundary:

$$\lim_{x\to x_0\in\partial\Omega}\int_{\partial\Omega}L(x,y)\,\theta_0(y)\,ds(y)=(1-d)\theta_0(x)+\int_{\partial\Omega}L(x_0,y)\,\theta_0(y)\,ds(y)$$

and $L(x,y)$ is weakly singular, $O(|x-y|^{2-d})$ for $x,y\in\partial\Omega$.
This gives for $x\in\partial\Omega$

$$(1+\kappa)\theta_0(x)+\kappa\int_{\partial\Omega}L(x,y)\,\theta_0(y)\,ds(y)=\operatorname{div}\mathsf{H}\boldsymbol{u}_0(x)\quad(x\in\partial\Omega)$$

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Lichtenstein's second kind integral equation

$$\frac{1+\kappa}{\kappa}\theta(x)+\int_{\partial\Omega}L(x,y)\,\theta(y)\,ds(y)=\frac{1}{\kappa}\operatorname{div}\mathrm{H}\boldsymbol{u}_0(x)\quad(x\in\partial\Omega)$$

Note : $\dfrac{1+\kappa}{\kappa}=\dfrac{2+\xi}{\xi}=1-2\sigma=\dfrac{\lambda+3\mu}{\lambda+\mu},\qquad\dfrac{1}{1+\kappa}=\dfrac{2\mu}{\lambda+3\mu}$

From Lichtenstein's original :

$$(20)\qquad\Theta(\bar\sigma)=\frac{2\mu}{5\lambda+7\mu}\Lambda(\bar\sigma)+\frac{\lambda+\mu}{4\pi(5\lambda+7\mu)}\int_S\bar\varrho\,\frac{\partial^2G}{\partial\bar\varrho\,\partial n}\,\Theta(\sigma)\,d\sigma.$$

$\lambda+3\mu\longleftrightarrow5\lambda+7\mu$: a little sign error in a jump relation...

Observation

The Cosserat eigenvalue problem for $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$

$$\sigma \Delta \boldsymbol{u} = \nabla \operatorname{div} \boldsymbol{u}$$

and the eigenvalue problem of the Schur complement operator $\mathscr{S}$ for $p \in L^2_\circ(\Omega)$

$$\mathscr{S} p = \sigma p$$

are equivalent .

Recall : $\mathscr{S} = \operatorname{div} \Delta^{-1} \nabla$

If $\boldsymbol{u}$ is a Cosserat eigenfunction, then $p = \operatorname{div} \boldsymbol{u}$ satisfies

$$\sigma p = \sigma \operatorname{div} \boldsymbol{u} = \operatorname{div} \Delta^{-1} \nabla \operatorname{div} \boldsymbol{u} = \mathscr{S} p.$$

Note : If $\operatorname{div} \boldsymbol{u} = 0$, then $\sigma \Delta \boldsymbol{u} = 0$, hence $\boldsymbol{u} = 0$ or $\sigma = 0$.
Conversely, if p is an eigenfunction of $\mathscr{S}$, then $\boldsymbol{u} = \Delta^{-1} \nabla p$ satisfies $\Delta \boldsymbol{u} = \nabla p$ and $\operatorname{div} \boldsymbol{u} = \mathscr{S} p = \sigma p$, hence

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Definition from now on

The Cosserat eigenvalue problem is the study of the spectrum of the bounded positive selfadjoint operator $\mathscr{S} = \operatorname{div} \Delta^{-1} \nabla$ in $L^2_\circ(\Omega)$.

Definition

The Cosserat constant of the domain Ω is

$$\sigma(\Omega) = \min \operatorname{Sp}(\mathscr{S})$$

This excludes the trivial eigenfunctions at $\sigma = 0$ satisfying $\operatorname{div} u = 0$ or $p =$const.

Example : For the ball $B_R(0)$ we have seen

$$\sigma(B_R(0)) = \frac{1}{d}.$$

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Why is $\mathscr{S}$ selfadjoint?

Define $\boldsymbol{w}(p) = \Delta^{-1}\nabla p$.
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$$\int_\Omega \nabla\boldsymbol{w} : \nabla\boldsymbol{v} = -\langle\nabla p, \boldsymbol{v}\rangle = \int_\Omega p \operatorname{div}\boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$$

For $p, q \in L^2_\circ(\Omega)$:

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A Lemma (integration by parts in C_0^∞)

For $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ there holds

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} = \int_\Omega \operatorname{div} \boldsymbol{v} \operatorname{div} \boldsymbol{w} + \int_\Omega \mathbf{curl}\, \boldsymbol{v} : \mathbf{curl}\, \boldsymbol{w}$$

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$$\int_\Omega p\,\mathscr{S}q = \int_\Omega p \operatorname{div}\boldsymbol{w}(q) = \int_\Omega \nabla\boldsymbol{w}(p) : \nabla\boldsymbol{w}(q)$$

This is symmetric and positive.
Also

$$|\mathscr{S}p|^2 = |\operatorname{div}\boldsymbol{w}(p)|^2 \le |\nabla\boldsymbol{w}(p)|^2 = \int_\Omega p\,\mathscr{S}p \le |p|\,|\mathscr{S}p|$$

Thus

$$\|\mathscr{S}\| \le 1 \qquad \text{and} \qquad \mathrm{Sp}(\mathscr{S}) \subset [0,1]\,.$$

Theorem (M. Crouzeix 1997)

Define

$$N = \Delta H_0^2(\Omega) = \{p \in L_\circ^2(\Omega) \mid p = \Delta q \text{ for some } q \in H_0^2(\Omega)\}$$

Then N is contained in the eigenspace of $\mathscr{S}$ for the eigenvalue $\sigma = 1$.
Split $L_\circ^2(\Omega)$ into the orthogonal sum

$$L_\circ^2(\Omega) = N \oplus M$$

If Ω is bounded and of class C^3 then $\mathscr{S} - \frac{1}{2}I : M \to M$ is compact, namely

$$\mathscr{S} - \frac{1}{2}I : \; M \to H^1(\Omega) \; \text{ bounded}$$

If $\Omega \subset \mathbb{R}^2$ has a corner, then $\mathscr{S} - \frac{1}{2}I : M \to M$ is not compact.

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Corollary

Mikhlin's Theorem is true for bounded C^3 domains

The Cosserat Spectrum according to Crouzeix: The space N

$$N = \Delta H^2_0(\Omega) = \{p \in L^2_\circ(\Omega) \mid p = \Delta q \text{ for some } q \in H^2_0(\Omega)\}$$

Let $p \in N$, $p = \Delta q$, $\boldsymbol{w} = \nabla q \in \boldsymbol{H}^1_0(\Omega)$.
Then

$$\Delta \boldsymbol{w} = \Delta \nabla q = \nabla \Delta q = \nabla p \Longrightarrow \boldsymbol{w} = \Delta^{-1} \nabla p$$

Hence $\operatorname{div} \boldsymbol{w} = \mathscr{S} p$.
On the other hand, $\operatorname{div} \boldsymbol{w} = \operatorname{div} \nabla q = \Delta q = p$.
Together this gives $\mathscr{S} p = p$, so p is an eigenfunction for $\sigma = 1$.

Note that $M = N^\perp$ is the space

$$M = \{p \in L^2_\circ(\Omega) \mid \int_\Omega p \Delta q = 0 \; \forall q \in H^2_0(\Omega)\} = \{p \in L^2_\circ(\Omega) \mid \Delta p = 0\}$$

(harmonic Bergman space $b^2(\Omega)$)

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$$M = \left\{p \in L^2_\circ(\Omega) \mid \int_\Omega p \Delta q = 0 \, \forall q \in H_0^2(\Omega)\right\} = \left\{p \in L^2_\circ(\Omega) \mid \Delta p = 0\right\}$$

(harmonic Bergman space $b^2(\Omega)$)

The Cosserat Spectrum according to Crouzeix: The space M

Here $\Omega \subset \mathbb{R}^3$, bounded, of class C^3.

Let $p \in M$. We can assume first that $p \in H^1(\Omega)$ (density!).
Choose $r \in C^3(\overline{\Omega})$ such that $r = 0$ and $\nabla r = \boldsymbol{n}$ on $\partial\Omega$, for example signed distance function of $\partial\Omega$. Define as usual $\boldsymbol{w} = \Delta^{-1}\nabla p$.
Note that here $\boldsymbol{w} \in H^2(\Omega)$. Trick : Set

$$u = \boldsymbol{w} \cdot \nabla r - \frac{1}{2} r p$$

$$\Rightarrow \Delta u = 2\nabla \boldsymbol{w} : \nabla\nabla r + \boldsymbol{w} \cdot \nabla\Delta r - \frac{1}{2}\Delta r\, p$$

We also know $u \in H^1_0(\Omega)$.
It follows that $u \in H^2(\Omega)$ and $|u|_2 \le C\,|\Delta u| \le C\,|p|$.
Let now $q = (\mathscr{S} - \frac{1}{2}I)p = \operatorname{div} \boldsymbol{w} - p/2$. $\Rightarrow q \in H^1(\Omega)$.
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Hence $|q|_1 \le C\,\|\gamma q\|_{H^{1/2}(\partial\Omega)} \le C\,|u|_2 \le C\,|p|$.

We have shown that for all $p \in M$: $\quad |(\mathscr{S} - \frac{1}{2}I)p|_1 \le C\,|p|$. Q.E.D.

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Crouzeix and Lichtenstein

Recall Lichtenstein's idea:

$\Delta p = 0$ & $\boldsymbol{w} = \Delta^{-1}\nabla p \Longrightarrow \Delta(\boldsymbol{w} - \frac{1}{2}\boldsymbol{x}p) = 0.$

Hence $\boldsymbol{w} - \frac{1}{2}\boldsymbol{x}p = \mathrm{H}\gamma(\boldsymbol{w} - \frac{1}{2}\boldsymbol{x}p) = -\frac{1}{2}\mathrm{H}(\boldsymbol{x}\gamma p)$,

H: harmonic extension and γ: boundary trace. Use $p = \mathrm{H}\gamma p$.

$\Longrightarrow \boldsymbol{w} = \frac{1}{2}(\boldsymbol{x}\mathrm{H}\gamma p - \mathrm{H}\boldsymbol{x}\gamma p)$.

$$\mathscr{S}p = \operatorname{div}\boldsymbol{w} = \frac{d}{2}p + \frac{1}{2}(\boldsymbol{x}\cdot\nabla\mathrm{H}\gamma p - \nabla\cdot\mathrm{H}\boldsymbol{x}\gamma p) = \frac{d}{2}p + \frac{1}{2}\mathscr{L}\gamma p$$

$\mathscr{L}$: Integral operator with Lichtenstein's kernel $L(x,y)$

$$\mathscr{L}\phi(x) = \int_{\partial\Omega}(\boldsymbol{x}-\boldsymbol{y})\cdot\nabla_x\partial_{n(y)}G(x,y)\phi(y)\,ds(y) \quad (x\in\Omega)$$

$$\gamma\mathscr{S}p = \frac{d}{2}\gamma p + \frac{1}{2}\gamma\mathscr{L}\gamma p = \frac{d}{2}\gamma p + \frac{1}{2}((1-d)\gamma p + L\gamma p) = \frac{1}{2}\gamma p + \frac{1}{2}L\gamma p$$

L: Boundary integral operator with Lichtenstein's kernel $L(x,y)$.

$$\gamma(\mathscr{S} - \tfrac{1}{2}I)\mathrm{H} = \tfrac{1}{2}L$$

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We have shown:

Theorem

The operator $\mathscr{S} - \frac{1}{2} I$ on the space of harmonic functions is equivalent to the weakly singular boundary integral operator $\frac{1}{2} L$ on the space of traces.

$\mathrm{H} : H^{-\frac{1}{2}}(\partial\Omega) \to b^2(\Omega)$ is an isomorphism with inverse γ.

$$\begin{array}{ccc}
b^2(\Omega) & \xrightarrow{\mathscr{S} - \frac{1}{2} I} & b^2(\Omega) \\
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A Simple Relation

$$\sigma(\Omega) = \beta(\Omega)^2$$

Proof : Rayleigh quotient: $\sigma(\Omega) = \min Sp(\mathscr{S}) = \min_{p \in L^2_\circ(\Omega)} \dfrac{\langle p, \mathscr{S} p \rangle}{\|p\|^2}$

As we have seen, with $w = w(p) = \Delta^{-1} \nabla p$,

$$\langle p, \mathscr{S} p \rangle = \langle \nabla w, \nabla w \rangle = \langle p, \operatorname{div} w \rangle.$$

Hence

$$\frac{\langle p, \mathscr{S} p \rangle}{\|p\|^2} = \left(\frac{\langle p, \operatorname{div} w \rangle}{\|p\| \, |w|_1} \right)^2$$

But for $v \in H^1_0(\Omega)$: $\quad \langle p, \operatorname{div} v \rangle = \langle \nabla w, \nabla v \rangle \le |w|_1 |v|_1 = \dfrac{\langle p, \operatorname{div} w \rangle}{|w|_1} |v|_1$

$$\Longrightarrow \quad \frac{\langle p, \operatorname{div} w \rangle}{\|p\| \, |w|_1} = \sup_{v \in H^1_0(\Omega)} \frac{\langle p, \operatorname{div} v \rangle}{\|p\| \, |v|_1}$$

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$$\Longrightarrow \quad \sigma(\Omega) = \beta(\Omega)^2$$

We have just seen that

$$\beta(\Omega) = \inf_{p \in L^2_\circ(\Omega)} \frac{\langle p, \operatorname{div} \boldsymbol{w}(p) \rangle}{|p| \, |\boldsymbol{w}(p)|_1}$$

where $\boldsymbol{w}(p) = \Delta^{-1} \nabla p$.

On the other hand, we have seen earlier that also

$$\beta(\Omega) = \inf_{p \in L^2_\circ(\Omega)} \frac{\langle p, \operatorname{div} \boldsymbol{v}(p) \rangle}{|p| \, |\boldsymbol{v}(p)|_1}$$

where $\boldsymbol{v}(p) = Bp$ with a minimal-norm right inverse B of the div operator. Such a right inverse can be obtained by observing that

$$\operatorname{div} : \big(\ker \operatorname{div}\big)^{\perp} \to L^2_\circ(\Omega)$$

is an isomorphism and taking for B its inverse.

For general $p \in L^2_\circ(\Omega)$, $\quad \dfrac{\langle p, \operatorname{div} \boldsymbol{w}(p) \rangle}{|\boldsymbol{w}(p)|_1} \neq \dfrac{\langle p, \operatorname{div} \boldsymbol{v}(p) \rangle}{|\boldsymbol{v}(p)|_1}$

Cosserat and LBB: A Curious Observation

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Question

For which $p \in L^2_\circ(\Omega)$ do these two quotients coincide?

Answer

For $p \in L^2_\circ(\Omega)$ one has

$$\frac{\langle p, \operatorname{div} \boldsymbol{w}(p)\rangle}{|\boldsymbol{w}(p)|_1} = \frac{\langle p, \operatorname{div} \boldsymbol{v}(p)\rangle}{|\boldsymbol{v}(p)|_1}$$

if and only if p is a Cosserat eigenfunction.

Proof :

$$\frac{\langle p, \operatorname{div} \boldsymbol{w}(p)\rangle}{|\boldsymbol{w}(p)|_1} = \frac{|\boldsymbol{w}(p)|_1^2}{|\boldsymbol{w}(p)|_1} = |\boldsymbol{w}(p)|_1 = \langle p, \mathscr{S} p\rangle^{\frac{1}{2}}$$

With $p = \mathscr{S} q$ we have $p = \operatorname{div} \boldsymbol{w}(q)$, hence $\boldsymbol{w}(q) = \boldsymbol{v}(p)$, hence

$$|\boldsymbol{v}(p)|_1 = |\boldsymbol{w}(q)|_1 = \langle q, \mathscr{S} q\rangle^{\frac{1}{2}} = \langle p, \mathscr{S}^{-1} p\rangle^{\frac{1}{2}}$$

$$\frac{\langle p, \operatorname{div} \boldsymbol{v}(p)\rangle}{|\boldsymbol{v}(p)|_1} = \frac{|p|^2}{|\boldsymbol{v}(p)|_1} = \frac{|p|^2}{\langle p, \mathscr{S}^{-1} p\rangle^{\frac{1}{2}}}$$

The question is therefore: When do we have

$$\langle p, \mathscr{S} p\rangle^{\frac{1}{2}} = \frac{|p|^2}{\langle p, \mathscr{S}^{-1} p\rangle^{\frac{1}{2}}}$$

$$\iff$$

$$|p|^2 = \langle p, \mathscr{S} p\rangle^{\frac{1}{2}} \langle p, \mathscr{S}^{-1} p\rangle^{\frac{1}{2}}$$

Now with Cauchy-Schwarz:

$$\begin{aligned} \|p\|^2 &= \langle \mathscr{S}^{1/2} p, \mathscr{S}^{-1/2} p\rangle \\ &\le \langle \mathscr{S}^{1/2} p, \mathscr{S}^{1/2} p\rangle^{\frac{1}{2}} \langle \mathscr{S}^{-1/2} p, \mathscr{S}^{-1/2} p\rangle^{\frac{1}{2}} \\ &= \langle p, \mathscr{S} p\rangle^{\frac{1}{2}} \langle p, \mathscr{S}^{-1} p\rangle^{\frac{1}{2}} \end{aligned}$$

Equality holds if and only if $\mathscr{S}^{1/2}p$ and $\mathscr{S}^{-1/2}p$ are proportional:

$$\mathscr{S}^{1/2} p = \sigma \mathscr{S}^{-1/2} p \quad \iff \quad \mathscr{S} p = \sigma p.$$

Cosserat and LBB: A Curious Observation

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Theorem (Friedrichs 1937, Horgan&Payne 1983)

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then

1. $\sigma(\Omega) > 0$ (Cosserat)
2. $\beta(\Omega) > 0$ (LBB)
3. $K(\Omega) < \infty$ (Korn)
4. $\Gamma(\Omega) < \infty$ (Friedrichs)

The following relations are true:

$$\frac{K(\Omega)}{2} = \frac{1}{\sigma(\Omega)} = \frac{1}{\beta(\Omega)^2} = \Gamma(\Omega) + 1$$

Remarks:

Friedrichs considers piecewise C^1 domains with no outgoing cusps.

Horgan&Payne consider simply connected Lipschitz domains, but make implicitly additional assumptions ($\sim C^2$).

Conjecture: For any bounded domain $\Omega \subset \mathbb{R}^2$, 1–4 are equivalent.

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Theorem [H&P...]

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain satisfying $\beta(\Omega) > 0$.
Then $\Gamma(\Omega) < \infty$ and

$$\Gamma(\Omega) \le \frac{1}{\beta(\Omega)^2} - 1\,.$$

Proof: Let $w = g + ih$ be a holomorphic function with $g \in L^2(\Omega)$ and $h \in L^2_\circ(\Omega)$. Then for any $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$ there holds

$$\int_\Omega h(x)\,\mathrm{div}\,\boldsymbol{u}(x)dx = -\langle \nabla h, \boldsymbol{u}\rangle = -\langle \mathbf{curl}\,g, \boldsymbol{u}\rangle = -\int_\Omega g(x)\cdot \mathrm{curl}\,\boldsymbol{u}(x)\,dx$$

Since $\beta(\Omega) > 0$, there exists $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$ such that

$$\mathrm{div}\,\boldsymbol{u} = h \quad \text{and} \quad |\,\mathrm{curl}\,\boldsymbol{u}|^2 = |\boldsymbol{u}|_1^2 - |\,\mathrm{div}\,\boldsymbol{u}|^2 \le \left(\tfrac{1}{\beta(\Omega)^2} - 1\right)|h|^2\,.$$

$$\text{Then} \qquad |h|^2 = \int_\Omega h(x)\,\mathrm{div}\,\boldsymbol{u}(x)dx = -\int_\Omega g(x)\cdot\mathrm{curl}\,\boldsymbol{u}(x)\,dx.$$

$$\Longrightarrow |h|^2 \le \sqrt{\tfrac{1}{\beta(\Omega)^2} - 1}\,|h|\,|g| \quad \Longrightarrow \quad |h|^2 \le \left(\tfrac{1}{\beta(\Omega)^2} - 1\right)|g|^2.$$

Theorem [Co&Da 2012]

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain satisfying $\Gamma(\Omega) < \infty$.
Then $\sigma(\Omega) > 0$ and

$$\sigma(\Omega) \geq \frac{1}{\Gamma(\Omega)+1}\,.$$

Proof: Let $p \in L^2_\circ(\Omega)$ and $\boldsymbol{u} = \Delta^{-1}\nabla p$. We have to show

$$\frac{|\boldsymbol{u}|_1^2}{|p|^2} \geq \frac{1}{\Gamma(\Omega)+1}$$

or equivalently

$$|p|^2 \leq (\Gamma(\Omega)+1)|\boldsymbol{u}|_1^2.$$

The variational formulation for $\boldsymbol{u}$ is

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) : (\boldsymbol{u}, \boldsymbol{v})_1 = (p, \operatorname{div} \boldsymbol{v})\,.$$

Define $q = \operatorname{div} \boldsymbol{u}$ and $g = \operatorname{curl} \boldsymbol{u}$ and observe successively:

Friedrichs $\Longrightarrow$ Cosserat

$(\boldsymbol{u},\boldsymbol{v})_1 = (p, \operatorname{div}\boldsymbol{v})$ and $q = \operatorname{div}\boldsymbol{u}$ and $g = \operatorname{curl}\boldsymbol{u} \Rightarrow$

$$\begin{aligned}
(p,q) &= |\boldsymbol{u}|_1^2 && = \|q\|^2 + \|g\|^2 \\
\Delta q &= \operatorname{div}\Delta\boldsymbol{u} && = \Delta p \\
\Delta g &= \operatorname{curl}\Delta\boldsymbol{u} && = 0 \\
\mathbf{curl}\, g - \nabla q &= -\Delta\boldsymbol{u} && = -\nabla p \\
\|g\|^2 &= (p,q) - \|q\|^2 && = (q, p-q)
\end{aligned}$$

It follows that $g \in L^2_\circ(\Omega)$ and $q - p \in L^2_\circ(\Omega)$ are conjugate harmonic functions. Friedrichs' inequality gives

$$\|p-q\|^2 \le \Gamma(\Omega)\,\|g\|^2 .$$

$$\|g\|^2 \le \|q\|\|p-q\| \le \|q\|\sqrt{\Gamma(\Omega)}\|g\|,$$
$$\|g\|^2 \le \Gamma(\Omega)\|q\|^2 .$$

Finally

$$\begin{aligned}
\|p\|^2 &= \|p-q\|^2 - \|q\|^2 + 2(p,q) \\
&= \|p-q\|^2 + \|g\|^2 + \|q\|^2 + \|g\|^2 \\
&\le \Gamma(\Omega)\|g\|^2 + \Gamma(\Omega)\|q\|^2 + \|q\|^2 + \|g\|^2 \\
&= (\Gamma(\Omega)+1)\,|\boldsymbol{u}|_1^2 .
\end{aligned}$$

$(\boldsymbol{u},\boldsymbol{v})_1 = (p, \operatorname{div}\boldsymbol{v})$ and $q = \operatorname{div}\boldsymbol{u}$ and $g = \operatorname{curl}\boldsymbol{u} \Rightarrow$

$$\begin{aligned} (p,q) &= |\boldsymbol{u}|_1^2 && = \|q\|^2 + \|g\|^2 \\ \Delta q &= \operatorname{div}\Delta\boldsymbol{u} && = \Delta p \\ \Delta g &= \operatorname{curl}\Delta\boldsymbol{u} && = 0 \\ \mathbf{curl}\, g - \nabla q &= -\Delta\boldsymbol{u} && = -\nabla p \\ \|g\|^2 &= (p,q) - \|q\|^2 && = (q, p-q) \end{aligned}$$

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Corollary [Co&Da 2012]

For any bounded domain Ω in $\mathbb{R}^2$, the Cosserat constant is positive if and only if the Friedrichs inequality holds, and

$$\frac{1}{\sigma(\Omega)} = \Gamma(\Omega) + 1 \,.$$

Recall :

- $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad 1 \le i,j \le d,$
- $r_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j - \partial_j u_i), \quad 1 \le i,j \le d,$

Definition

Let Ω be a domain in $\mathbb{R}^d$. It is said to satisfy the second Korn inequality if there exists a positive constant K such that for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisfying the condition

$$\int_\Omega r_{ij}(\boldsymbol{u})(x)\,dx = 0, \quad 1 \le i,j \le d$$

there holds the estimate

$$\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} \le K \|\boldsymbol{e}(\boldsymbol{u})\|^2_{L^2(\Omega)}$$

If such a K exists we denote by $K(\Omega)$ the smallest such K.

Proof of Korn's second inequality

Theorem

Let $\Omega \subset \mathbb{R}^d$ *be such that* $\beta(\Omega) > 0$. *Then* $K(\Omega) < \infty$ *and*

$$K(\Omega)^2 \le 1 + \frac{2(d-1)}{\beta(\Omega)^2}.$$

For the proof, one applies the equivalent definition of $\beta(\Omega)$

$$\forall p \in L^2_0(\Omega) \;:\; \beta |p| \le \|\nabla p\|_{\boldsymbol{H}^{-1}(\Omega)}$$

to the functions $r_{ij} \in L^2_0(\Omega)$.

$$\text{Trick :} \quad \partial_k r_{ij} = \partial_i e_{jk} - \partial_j e_{ik}$$

$$\Rightarrow \quad |r(\boldsymbol{u})|^2 \le \cdots \le \frac{2(d-1)}{\beta^2} |e(\boldsymbol{u})|^2$$

$$|\nabla \boldsymbol{u}|^2 = |e(\boldsymbol{u})|^2 + |r(\boldsymbol{u})|^2 \le \big(1 + \frac{2(d-1)}{\beta^2}\big) |e(\boldsymbol{u})|^2.$$

Part III

Various Kinds of Domains

Unions of domains

Union with overlap

Let the domain $\Omega \subset \mathbb{R}^d$ satisfy $\Omega = \Omega_1 \cup \Omega_2$ with $\sigma(\Omega_j) > 0$ $(j = 1, 2)$.
Then $\sigma(\Omega) > 0$.
Quantitative estimates by Dafermos (1968, for Korn), Galdi (1994).

Union without overlap

Let the domain $\Omega \subset \mathbb{R}^d$ satisfy $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$, with $\sigma(\Omega_j) > 0$ $(j = 1, 2)$.
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Caution : No estimate for $\sigma(\Omega)$ possible depending only on Ω_1 and Ω_2.

Examples in $\mathbb{R}^2$:

$$\Omega_0 = (0, L) \times (-\ell, \ell), \quad \Omega = \Omega_0 \cup B_\ell(0,0) \cup B_\ell(L,0) \quad \Longrightarrow \quad \sigma(\Omega) \geq \frac{\sigma(\Omega_0)}{89}$$

$$\Omega_\varepsilon = B_1(0) \setminus (-1, 1-\varepsilon) \times \{0\} \quad \Longrightarrow \quad \sigma(\Omega_\varepsilon) = O(\varepsilon^2)$$

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Theorem (Bogovskiĭ 1979, Galdi 1994)

Let $\Omega \subset \mathbb{R}^n$ be starshaped with respect to a ball B. There exists a constant γ_d only depending on the dimension d such that

$$\sigma(\Omega) \geq \gamma_d \left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(\Omega)} \right)^{2d+2}$$

Let Ω be a finite union of bounded starshaped domains.

Then $\sigma(\Omega) > 0$.

This includes all bounded Lipschitz domains, possibly with cracks.

In dimension $d = 2$, we can prove

$$\sigma(\Omega) \geq \frac{1}{4} \left(\frac{\rho}{R} \right)^2$$

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Bogovskiĭ's integral operator

Let $\Omega \subset \mathbb{R}^d$ be starshaped with respect to a ball B and $\omega \in C_0^\infty(B)$ be such that $\int \omega = 1$.

Define $\mathbf{T}p(x) = \int_\Omega \mathbf{G}(x,y)p(y)\,dy$ with

$$\mathbf{G}(x,y) = \frac{\boldsymbol{x}-\boldsymbol{y}}{|x-y|^d}\int_{|x-y|}^\infty \omega\Big(y + t\frac{x-y}{|x-y|}\Big)\, t^{d-1}\,dt$$

Then $\mathbf{T} : L_0^2(\Omega) \to \boldsymbol{H}_0^1(\Omega)$ is continuous and $\operatorname{div}\mathbf{T}p = p$ (right inverse!).

Explanation :

The adjoint operator T' is the regularized Poincaré path integral

$$\mathrm{T}'\boldsymbol{u}(x) = \int_B \omega(a)\int_a^x \boldsymbol{u}\cdot\mathbf{ds}\,da = \int_B \omega(a)(\boldsymbol{x}-\boldsymbol{a})\cdot\int_0^1 \boldsymbol{u}(a+t(x-a))\,dt\,da$$

satisfying $\mathrm{T}'\nabla p(x) = p(x) - \int_B p(a)\,\omega(a)\,da$ (left inverse on $L^2(\Omega)/\mathbb{R}$)

Lemma (Co&McIntosh 2010)

T and T' are pseudodifferential operators on $\mathbb{R}^d$ of order -1.

$\forall s \in \mathbb{R}$: $\mathbf{T} : \widetilde{H}^s(\Omega) \to \widetilde{\boldsymbol{H}}^{s+1}(\Omega)$ and $\mathrm{T}' : \boldsymbol{H}^s(\Omega) \to H^{s+1}(\Omega)$

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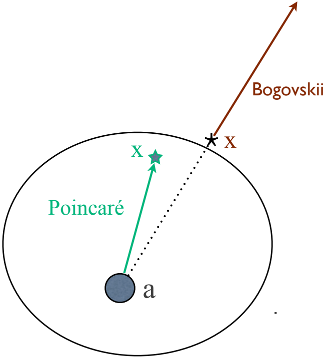

Support properties:

- For $x \in \Omega$, $T'\boldsymbol{u}(x)$ depends only on $\boldsymbol{u}\big|_{\Omega}$
- If $p = 0$ on $\mathbb{R}^d \setminus \Omega$, then $\mathbf{T}p = 0$ on $\mathbb{R}^d \setminus \Omega$.

Corners and Essential Spectrum (in $\mathbb{R}^2$)

Mellin transform technique (Kondrat'ev 1967) for the Lamé operator

$$A_\sigma = -\sigma\Delta + \nabla \operatorname{div}$$

Singularities of the form $r^\lambda \phi(\theta)$.
Characteristic equation for a corner of opening ω:

$$(*) \qquad (1-2\sigma)\frac{\sin\lambda\omega}{\lambda} = \pm \sin\omega.$$

Theorem

For $\sigma \in [0,1] \setminus \{0, \frac{1}{2}, 1\}$, A_σ is Fredholm iff the equation $(*)$ has no solution on the line $\Re e\,\lambda = 0$.

With $z = \lambda\omega$, we rewrite $(*)$:

$$(1-2\sigma)\frac{\sin z}{z} = \pm\frac{\sin\omega}{\omega}.$$

Result :

- $(*)$ has roots on the line $\Re e\,\lambda = 0$ iff $|1-2\sigma|\omega \le |\sin\omega|$
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Theorem [Co & Dauge 2000]

Ω piecewise smooth with corners of opening ω_j.

$$\mathrm{Sp}_{\mathrm{ess}}(\mathscr{S}) = \bigcup_{\text{corners } j} \left[\tfrac{1}{2} - \frac{|\sin\omega_j|}{2\omega_j}, \tfrac{1}{2} + \frac{|\sin\omega_j|}{2\omega_j}\right] \cup \{1\}$$

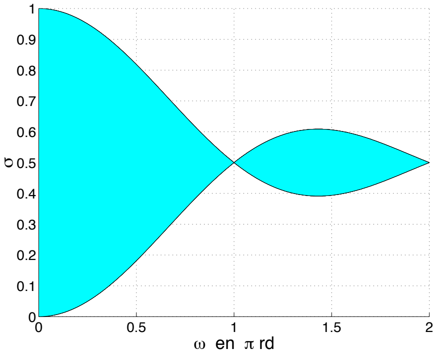
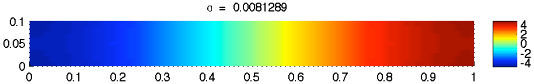


Figure: Essential spectrum: σ vs. opening ω

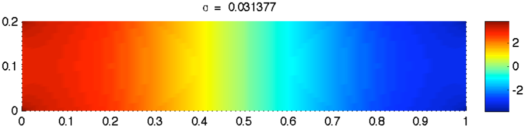
Example : Rectangle:

$$\mathrm{Sp}_{\mathrm{ess}}(\mathscr{S}\big|_M) = [\tfrac{1}{2} - \tfrac{1}{\pi}, \tfrac{1}{2} + \tfrac{1}{\pi}]$$
$$= [0.181, 0.818]$$

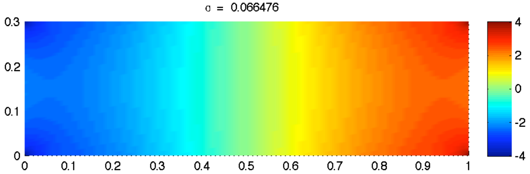
First eigenfunction for the rectangle



Rectangle: $[0,1] \times [0,0.1]$
$\sigma_{\text{approx}} = 0.0081$

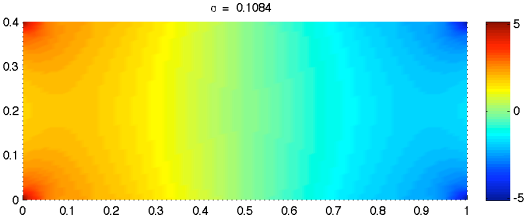


Rectangle: $[0,1] \times [0,0.2]$
$\sigma_{\text{approx}} = 0.0314$



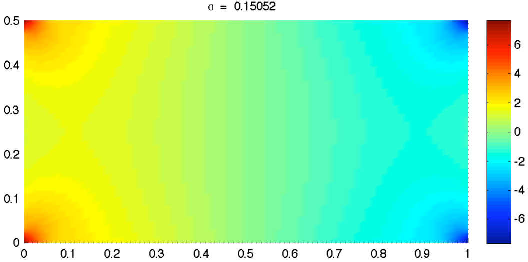
Rectangle: $[0,1] \times [0,0.3]$
$\sigma_{\text{approx}} = 0.0665$

First eigenfunction for the rectangle



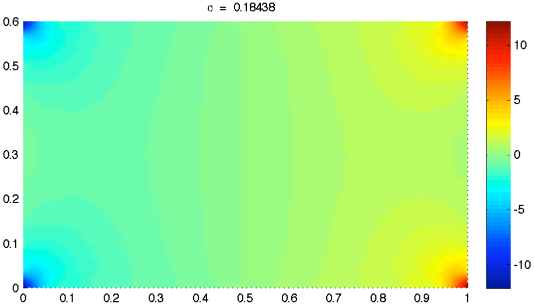
Rectangle: $[0,1] \times [0,0.4]$
$\sigma_{\text{approx}} = 0.1084$

First eigenfunction for the rectangle



Rectangle: $[0,1] \times [0,0.5]$
$\sigma_{\text{approx}} = 0.1505$

First eigenfunction for the rectangle



Rectangle: $[0,1] \times [0,0.6]$
$\sigma_{\text{approx}} = 0.1844$. (In the essential spectrum!).

Theorem (Horgan & Payne 1983)

Let $\Omega \subset \mathbb{R}^2$ be starshaped with respect to 0. For $x \in \partial\Omega$, let $\gamma(x) \in [0, \frac{\pi}{2})$ be the angle between x and the normal vector $\boldsymbol{n}(x)$: $\gamma(x) = \arccos \frac{\boldsymbol{x}\cdot\boldsymbol{n}(x)}{|\boldsymbol{x}|}$, and

$$\gamma = \gamma(\Omega) = \max_{x\in\partial\Omega} \gamma(x).$$

Then

$$\sigma(\Omega) \geq \frac{1-\sin\gamma}{2}$$

Square : $\gamma(\Omega) = \frac{\pi}{4} \implies \sigma(\Omega) \geq \frac{1}{2} - \frac{\sqrt{2}}{4} \approx 0.1464$

Regular Polygon : $\gamma(\Omega) = \frac{\pi}{n} \implies \sigma(\Omega) \geq \frac{1}{2} - \frac{\sin\pi}{n}$

Rectangle $(0,1)\times(0,\varepsilon)$: $\gamma(\Omega) = \frac{\pi}{2} - \arctan\varepsilon \Rightarrow \sigma(\Omega) \geq \frac{\varepsilon^2}{4} + O(\varepsilon^4)$

Compare with Ellipse $x^2 + \frac{y^2}{\varepsilon^2} = 1$ (Cosserats): $\sigma(\Omega) = \frac{\varepsilon^2}{1+\varepsilon^2}$

~~Theorem~~Conjecture (Horgan & Payne 1983)

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Theorem [Co&Da 2012, Monique's talk]

The Conjecture is proved for rectangles, triangles and regular polygons.

The Horgan–Payne Angle

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Theorem [Co&Da 2012, Monique's talk]

The Conjecture is proved for rectangles, triangles and regular polygons.

Small Cuts

Theorem (Co 2011)

Let $\Omega \subset \mathbb{R}^2$ be decomposed as

$$\Omega = \Omega^- \dot\cup \Gamma \dot\cup \Omega^+$$

where Γ is a straight segment of length L.
Then

$$\sigma(\Omega) \le \frac{8}{3} \frac{|\Omega|}{|\Omega^-|\,|\Omega^+|} L^2$$

Example : $\Omega_\varepsilon = B_1(0) \setminus (-1, 1-\varepsilon) \times \{0\} \implies \sigma(\Omega_\varepsilon) \le \frac{32}{3}\varepsilon^2$

Idea of proof:
Estimate $\int_\Omega q \operatorname{div} v$ for a piecewise constant function q.
One has to estimate the integral of $n \cdot v$ on the cut by the H^1 norm of v.
Explicit L^1 trace lemma in H^1_0:

$$\|u\|^2_{L^1([0,L])} \le \frac{8L^2}{3} \|\nabla u\|^2_{L^2(\mathbb{R}^2_+)} \qquad \forall u \in C_0^\infty(\mathbb{R}^2)$$

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Idea of proof:
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One has to estimate the integral of $\boldsymbol{n} \cdot \boldsymbol{v}$ on the cut by the H^1 norm of $\boldsymbol{v}$.
Explicit L^1 trace lemma in H^1_0:

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Corollary (Friedrichs 1937)

Let $\Omega \subset \mathbb{R}^2$ have an outward cusp. Then $\sigma(\Omega) = 0$.

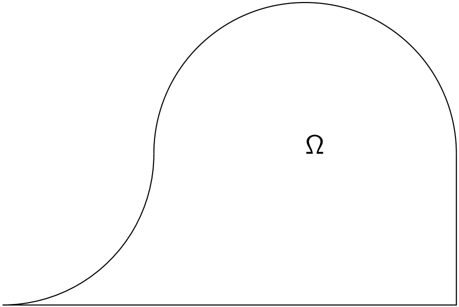

Figure: A domain with an external cusp

Thin Domains according to Chizhonkov&Olshanskii (2000)

Thin Rectangles

Let $\Omega = (0,1) \times (0,\varepsilon)$, $0 < \varepsilon \leq 1$. Then

$$\frac{\varepsilon^2}{60} \leq \sigma(\Omega) \leq \frac{\pi^2 \varepsilon^2}{12}$$

Thin Rings

Let $\Omega = \{x \in \mathbb{R}^2 \mid 1 < |x| < 1 + \varepsilon)$. Then with $s = 1 + \varepsilon$

$$\sigma(\Omega) = \frac{1}{2}\left(1 - \sqrt{\frac{s^2 - 1}{s^2 + 1} \frac{1}{\log s}}\right) \sim \frac{\varepsilon^2}{12}$$

Let $\Omega = (0,\pi) \times (-\rho,\rho)$. Aspect ratio $\varepsilon = \frac{2\rho}{\pi}$.

An explicit upper bound (Co&Dauge)

$$\sigma(\Omega) \le 1 - \frac{\sinh\rho}{\rho\cosh\rho}$$

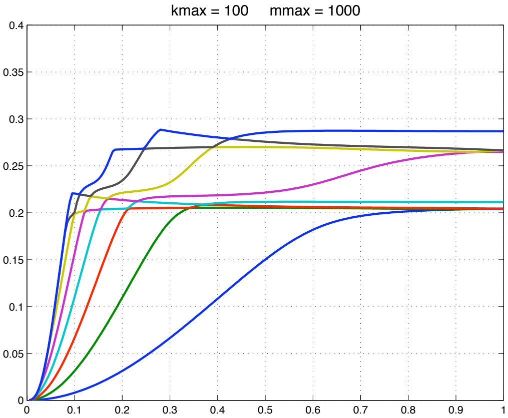

Figure: 8 lowest eigenvalues σ_ℓ of rectangle vs. aspect ratio ε

Comparison with upper and lower bounds

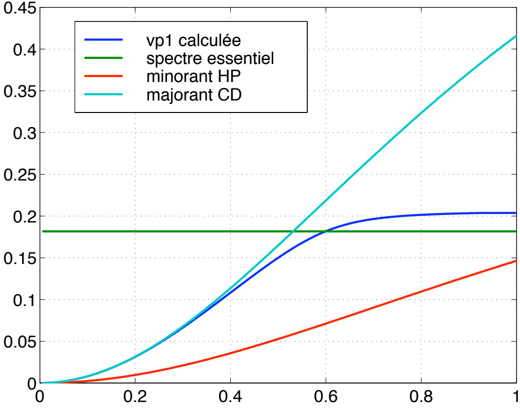

Figure: First eigenvalue σ_1 of rectangle vs. aspect ratio ε

Comparison with upper and lower bounds

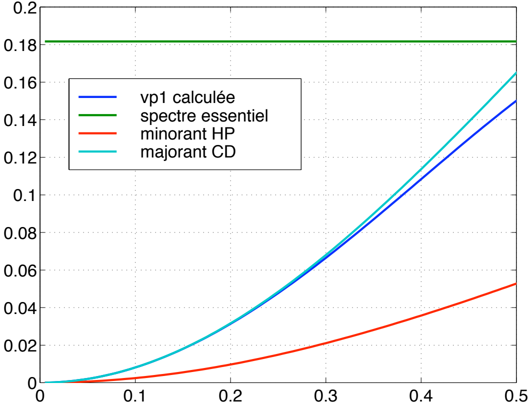

Figure: First eigenvalue σ_1 of rectangle vs. aspect ratio ε (zoom)

Comparison with upper and lower bounds

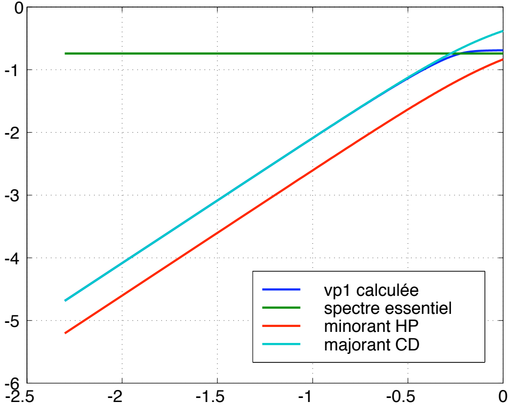

Figure: First eigenvalue σ_1 of rectangle vs. aspect ratio ε (log scale)

Definition

A domain $\Omega \subset \mathbb{R}^d$ with a distinguished point $\boldsymbol{x}_0$ is called a John domain if it satisfies the following “twisted cone” condition:
There exists a constant $\delta > 0$ such that, for any $\boldsymbol{y}$ in Ω, there is a rectifiable curve γ: $[0,\ell] \to \Omega$ parametrized by arclength such that

$$\gamma(0) = \boldsymbol{y}, \quad \gamma(\ell) = \boldsymbol{x}_0, \quad \text{and} \quad \forall t \in [0,\ell] : \; \operatorname{dist}(\gamma(t), \partial\Omega) \geq \delta\, t\,.$$

Here $\operatorname{dist}(\gamma(t), \partial\Omega)$ denotes the distance of $\gamma(t)$ to the boundary $\partial\Omega$.

Example : Every weakly Lipschitz domain is a John domain.

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Example : Every weakly Lipschitz domain is a John domain.

Figure: A weakly Lipschitz domain: the self-similar zigzag

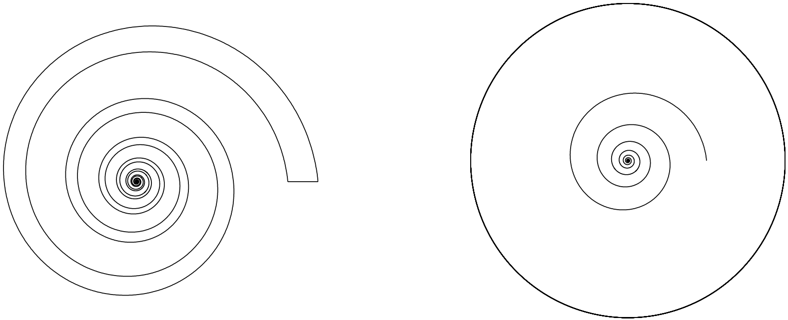


Figure: Weakly Lipschitz (left), John domain (right)

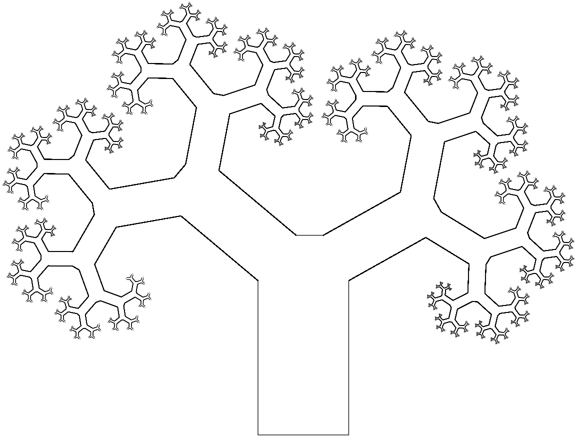


Figure: A John domain: the infinite tree

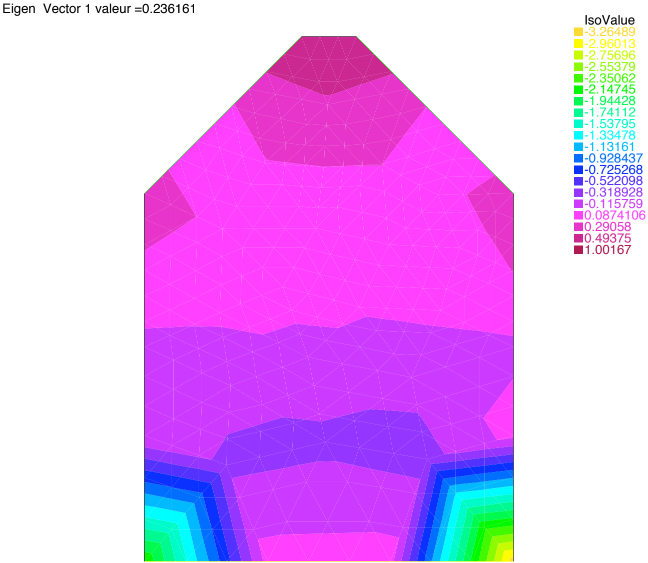
A Recent Result

Theorem (Acosta – Durán – Muschietti 2006)

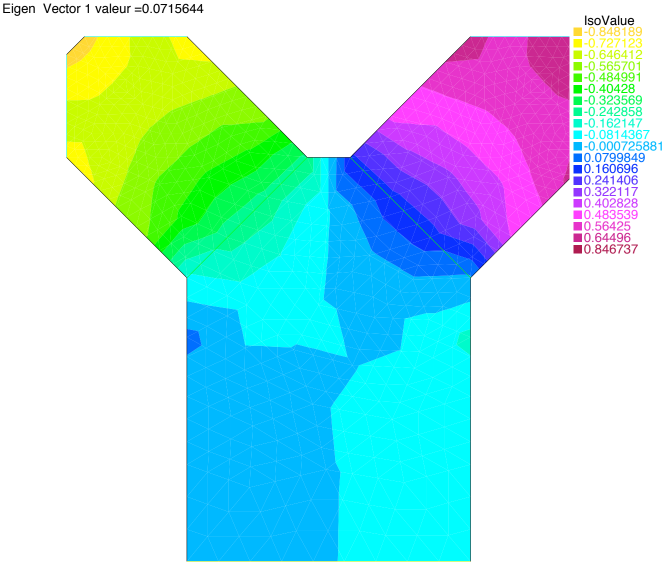
Let Ω be a John domain. Then $\sigma(\Omega) > 0$.

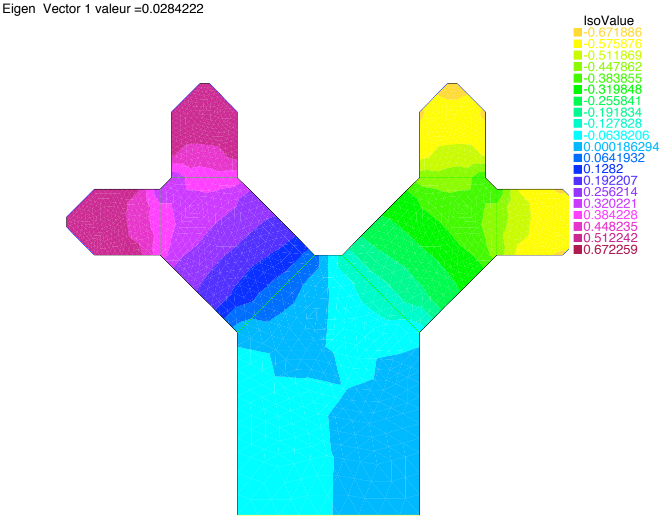
Thank you for your attention!

F. Hecht: Calculs poumon – Eigenvalue 1

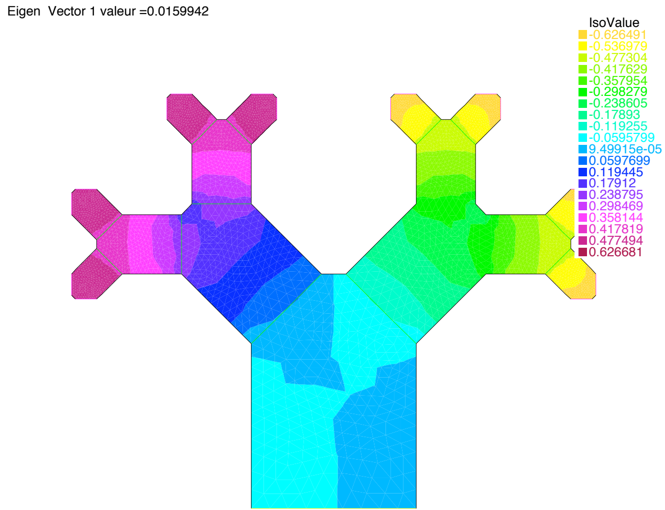


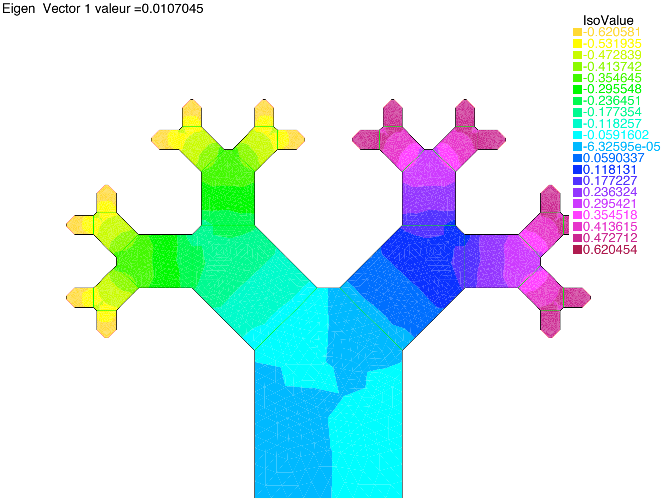
F. Hecht: Calculs poumon – Eigenvalue 1




Eigen Vector 1 valeur =0.0284222
IsoValue
-0.671886
-0.575876
-0.511869
-0.447862
-0.383855
-0.319848
-0.255841
-0.191834
-0.127828
-0.0638206
0.000186294
0.0641932
0.1282
0.192207
0.256214
0.320221
0.384228
0.448235
0.512242
0.672259

F. Hecht: Calculs poumon – Eigenvalue 1




Eigen Vector 1 valeur =0.0107045
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-0.531935
-0.472839
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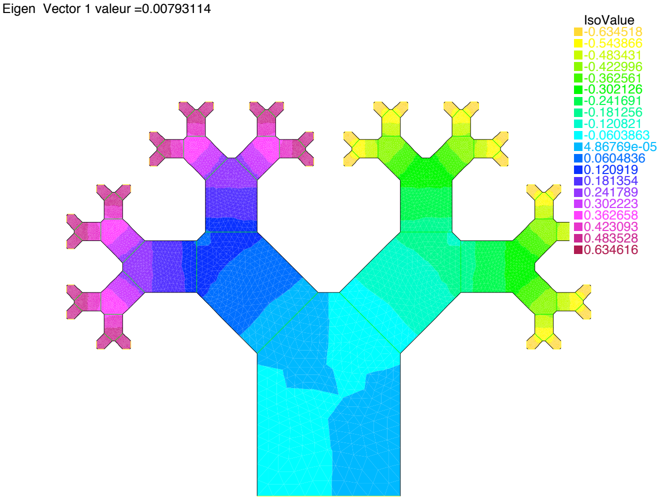
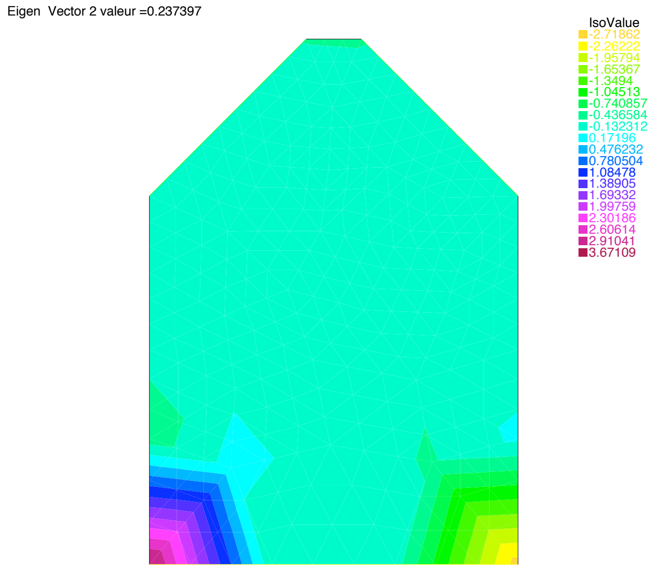
-0.354645
-0.295548
-0.236451
-0.177354
-0.118257
-0.0591602
-6.32595e-05
0.0590337
0.118131
0.177227
0.236324
0.295421
0.354518
0.413615
0.472712
0.620454

F. Hecht: Calculs poumon – Eigenvalue 1




Eigen Vector 2 valeur =0.237397
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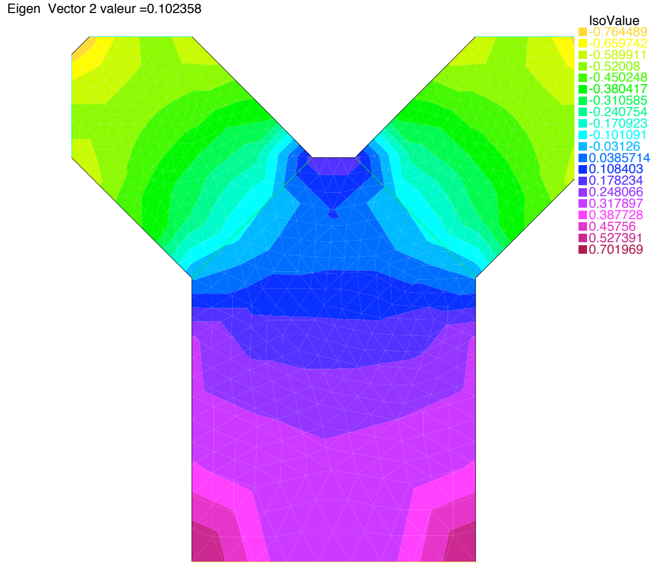
-2.26222
-1.95794
-1.65367
-1.3494
-1.04513
-0.740857
-0.436584
-0.132312
0.17196
0.476232
0.780504
1.08478
1.38905
1.69332
1.99759
2.30186
2.60614
2.91041
3.67109


Eigen Vector 2 valeur =0.102358
IsoValue
-0.764489
-0.659742
-0.589911
-0.52008
-0.450248
-0.380417
-0.310585
-0.240754
-0.170923
-0.101091
-0.03126
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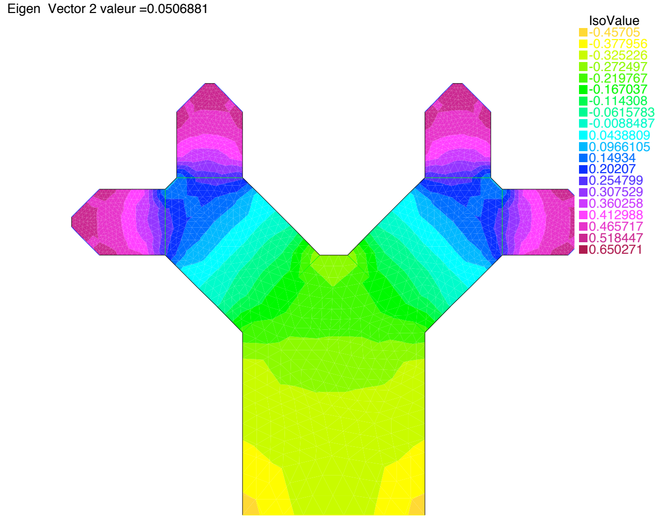
0.108403
0.178234
0.248066
0.317897
0.387728
0.45756
0.527391
0.701969


Eigen Vector 2 valeur =0.0506881
IsoValue
-0.45705
-0.377956
-0.325226
-0.272497
-0.219767
-0.167037
-0.114308
-0.0615783
-0.0088487
0.0438809
0.0966105
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0.20207
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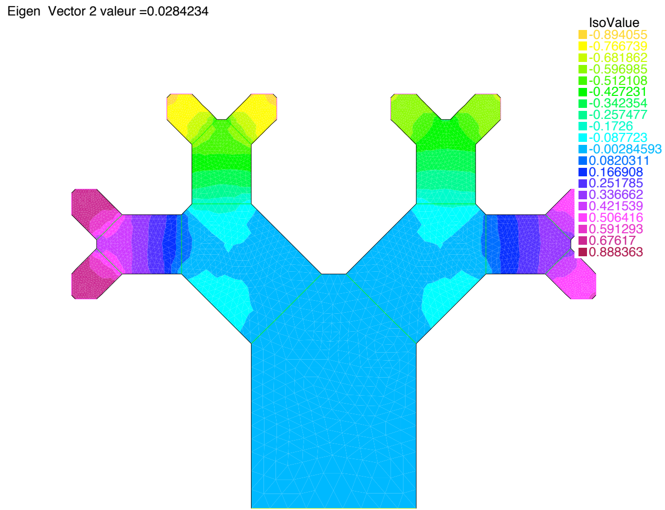
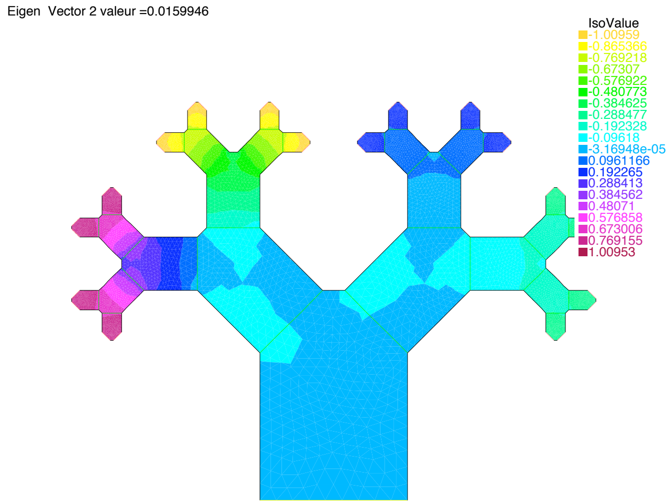
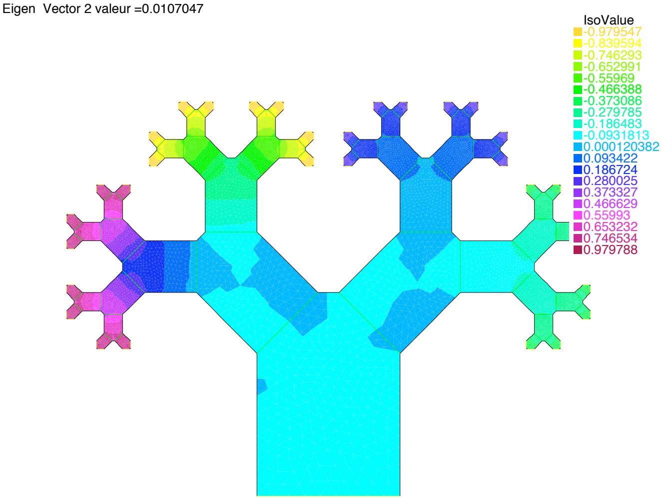
0.360258
0.412988
0.465717
0.518447
0.650271

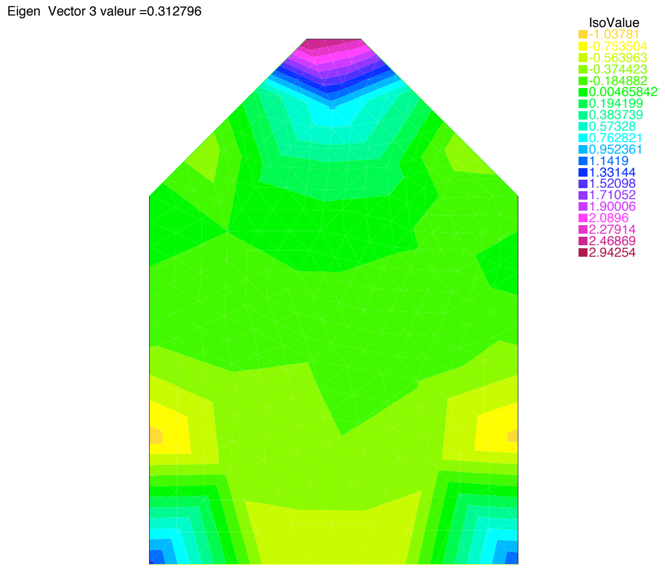

Eigen Vector 2 valeur =0.0284234
IsoValue
-0.894055
-0.766739
-0.681862
-0.596985
-0.512108
-0.427231
-0.342354
-0.257477
-0.1726
-0.087723
-0.00284593
0.0820311
0.166908
0.251785
0.336662
0.421539
0.506416
0.591293
0.67617
0.888363

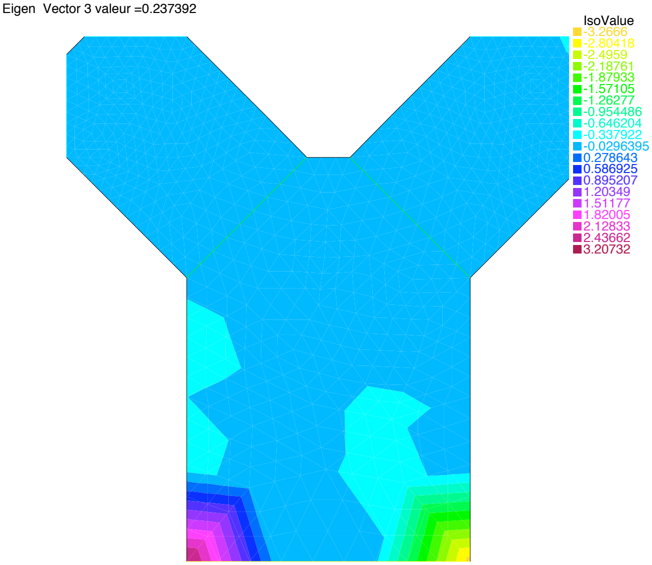
F. Hecht: Calculs poumon – Eigenvalue 2

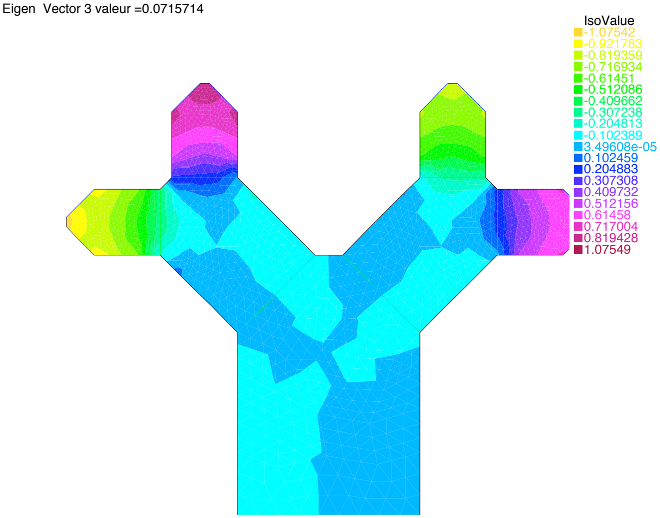


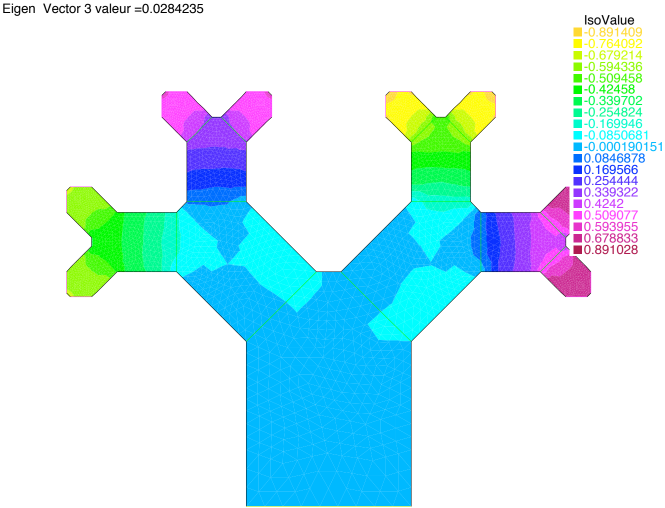

Eigen Vector 2 valeur =0.0107047
IsoValue
-0.979547
-0.839594
-0.746293
-0.652991
-0.55969
-0.466388
-0.373086
-0.279785
-0.186483
-0.0931813
0.000120382
0.093422
0.186724
0.280025
0.373327
0.466629
0.55993
0.653232
0.746534
0.979788

F. Hecht: Calculs poumon – Eigenvalue 3




Eigen Vector 3 valeur =0.237392
IsoValue
-3.2666
-2.80418
-2.4959
-2.18761
-1.87933
-1.57105
-1.26277
-0.954486
-0.646204
-0.337922
-0.0296395
0.278643
0.586925
0.895207
1.20349
1.51177
1.82005
2.12833
2.43662
3.20732


Eigen Vector 3 valeur =0.0715714
IsoValue
-1.07542
-0.921783
-0.819359
-0.716934
-0.61451
-0.512086
-0.409662
-0.307238
-0.204813
-0.102389
3.49608e-05
0.102459
0.204883
0.307308
0.409732
0.512156
0.61458
0.717004
0.819428
1.07549


Eigen Vector 3 valeur =0.0284235
IsoValue
-0.891409
-0.764092
-0.679214
-0.594336
-0.509458
-0.42458
-0.339702
-0.254824
-0.169946
-0.0850681
-0.000190151
0.0846878
0.169566
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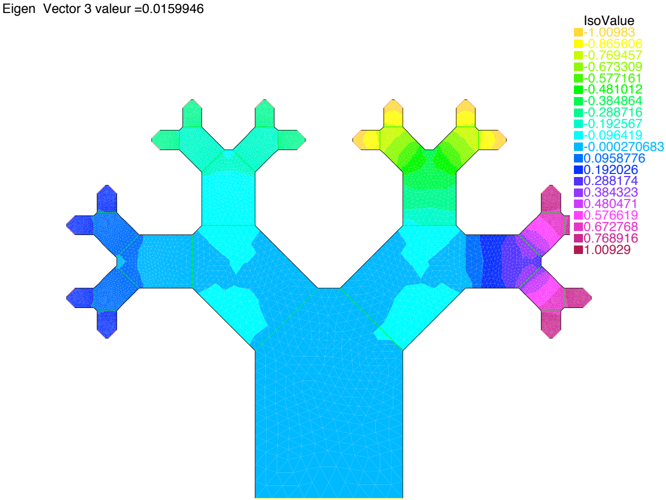
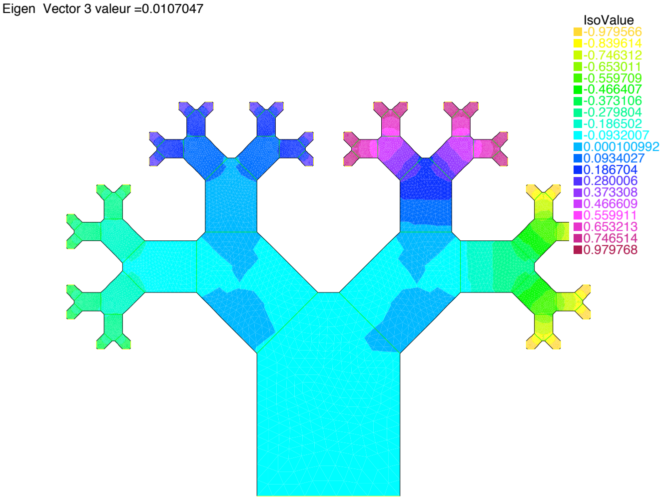
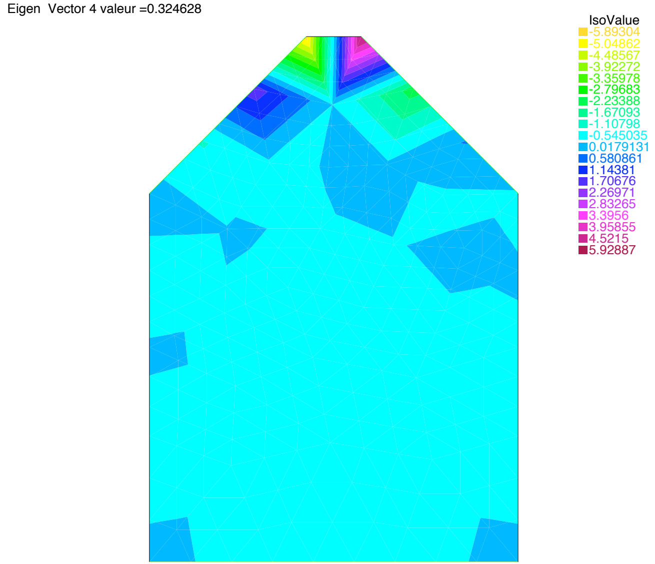
0.339322
0.4242
0.509077
0.593955
0.678833
0.891028

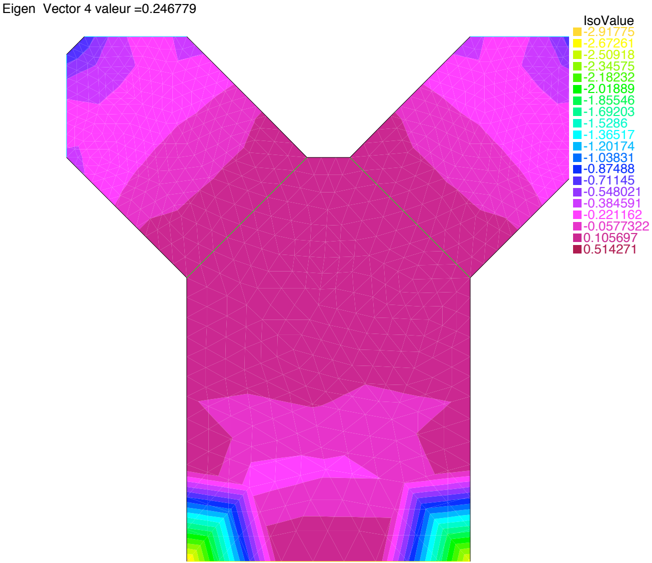
F. Hecht: Calculs poumon – Eigenvalue 3

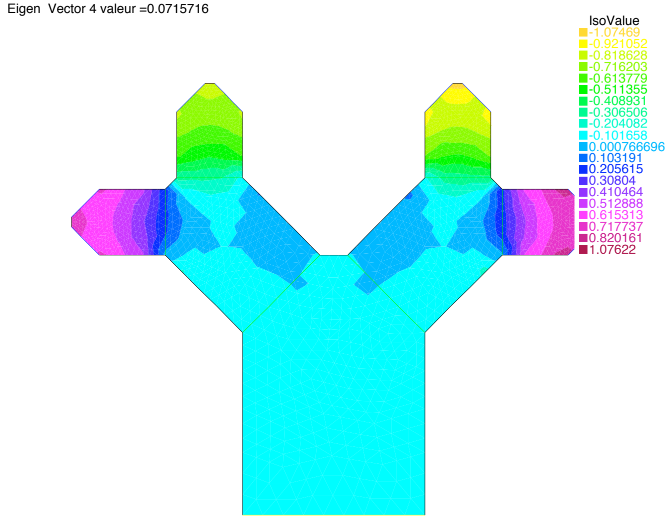


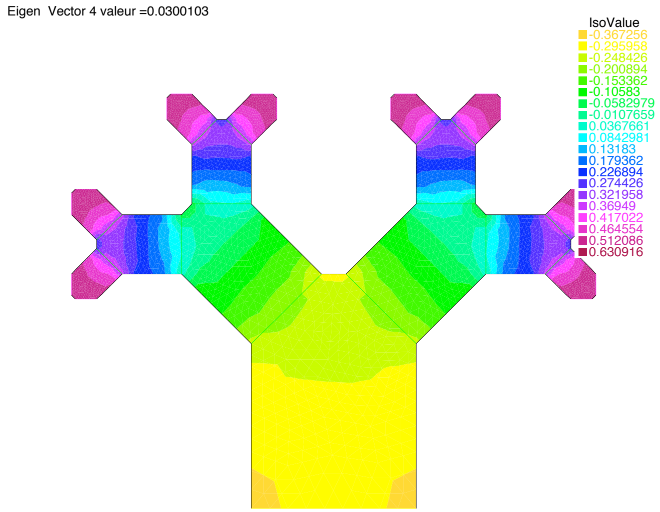
F. Hecht: Calculs poumon – Eigenvalue 3



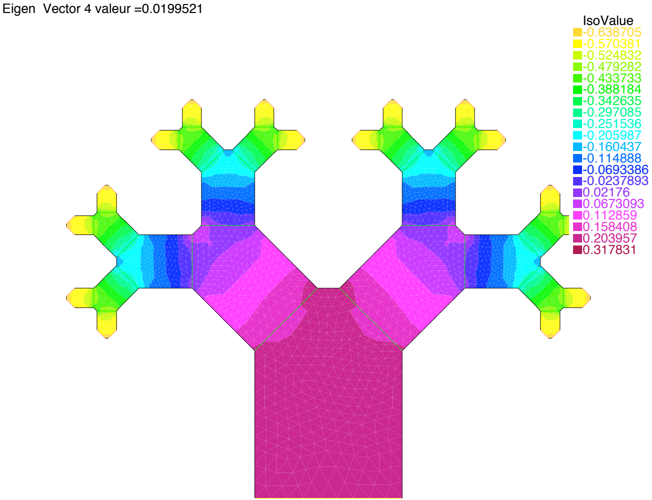

Eigen Vector 4 valeur =0.324628
IsoValue
-5.89304
-5.04862
-4.48567
-3.92272
-3.35978
-2.79683
-2.23388
-1.67093
-1.10798
-0.545035
0.0179131
0.580861
1.14381
1.70676
2.26971
2.83265
3.3956
3.95855
4.5215
5.92887


Eigen Vector 4 valeur =0.246779
IsoValue
-2.91775
-2.67261
-2.50918
-2.34575
-2.18232
-2.01889
-1.85546
-1.69203
-1.5286
-1.36517
-1.20174
-1.03831
-0.87488
-0.71145
-0.548021
-0.384591
-0.221162
-0.0577322
0.105697
0.514271


Eigen Vector 4 valeur =0.0715716
IsoValue
-1.07469
-0.921052
-0.818628
-0.716203
-0.613779
-0.511355
-0.408931
-0.306506
-0.204082
-0.101658
0.000766696
0.103191
0.205615
0.30804
0.410464
0.512888
0.615313
0.717737
0.820161
1.07622


Eigen Vector 4 valeur =0.0300103
IsoValue
-0.367256
-0.295958
-0.248426
-0.200894
-0.153362
-0.10583
-0.0582979
-0.0107659
0.0367661
0.0842981
0.13183
0.179362
0.226894
0.274426
0.321958
0.36949
0.417022
0.464554
0.512086
0.630916

F. Hecht: Calculs poumon – Eigenvalue 4



F. Hecht: Calculs poumon – Eigenvalue 4