Existence results for the heterogeneous Maxwell equations with different boundary conditions

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Outline of the talk

Perfectly conducting bc

We consider (non-stationary) Maxwell's equations:

$$\left\{\begin{array}{l} \varepsilon \frac{\partial E}{\partial t} - \mathbf{curl}\, H = 0 \text{ in } Q = \Omega \times]0, T[, \\ \mu \frac{\partial H}{\partial t} + \mathbf{curl}\, E = 0 \text{ in } Q, \\ \operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0 \text{ in } Q, \\ E \times \nu = 0,\ H \cdot \nu = 0 \text{ on } \Sigma = \Gamma \times]0, T[, \\ E(0) = E_0, \quad H(0) = H_0 \text{ in } \Omega, \end{array}\right. \tag{1}$$

where ν denotes the unit outer normal vector on Γ. This means that we suppose that the time evolution of the electric field E and magnetic field H is only driven by some initial data.
The bc are called perfectly conducting bc.

Magnetic boundary condition

Similarly we can consider (non-stationary) Maxwell's equations:

$$\left\{\begin{array}{l} \varepsilon\frac{\partial E}{\partial t} - \textbf{curl}\, H = 0 \text{ in } Q = \Omega\times]0,T[,\\ \mu\frac{\partial H}{\partial t} + \textbf{curl}\, E = 0 \text{ in } Q,\\ \operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0 \text{ in } Q,\\ H\times\nu = 0,\ E\cdot\nu = 0 \text{ on } \Sigma = \Gamma\times]0,T[,\\ E(0) = E_0, \quad H(0) = H_0 \text{ in } \Omega. \end{array}\right. \tag{2}$$

This pb is the adjoint of the previous one once we exchange the rule of ε and μ.

Assumptions on the domain and on the coefficients

Ω is a bounded, simply connected domain with a Lipschitz boundary Γ.

ε and μ are piecewise constant on Lipschitz polyhedral subdomains, in the sense that we assume that there exists a partition $\mathcal{P}$ of Ω in a finite set of Lipschitz polyhedra $\Omega_1, \cdots, \Omega_J$ such that on each Ω_j, $\varepsilon = \varepsilon_j$ and $\mu = \mu_j$, where ε_j and μ_j are positive constants.

A Lipschitz polyhedron is a bounded, simply connected Lipschitz domain with piecewise plane boundary.

Outline of the talk

Definitions

$$
\begin{aligned}
H(\mathrm{div}\varepsilon 0, \Omega) &= \{\chi \in L^2(\Omega)^3 | \mathrm{div}(\varepsilon\chi) = 0\},\\
H_0(\mathrm{div}\varepsilon 0, \Omega) &= \{\chi \in H(\mathrm{div}\varepsilon 0, \Omega) | \chi \cdot \nu = 0 \text{ on } \Gamma\},\\
H(\mathbf{curl}, \Omega) &= \{\chi \in L^2(\Omega)^3 | \mathbf{curl}\, \chi \in L^2(\Omega)^3\},\\
H_0(\mathbf{curl}, \Omega) &= \{\chi \in H(\mathbf{curl}, \Omega) | \chi \times \nu = 0 \text{ on } \Gamma\},\\
X_T^0(\Omega, \mu) &= H_0(\mathrm{div}\mu 0, \Omega) \cap H(\mathbf{curl}, \Omega),\\
X_N^0(\Omega, \varepsilon) &= H(\mathrm{div}\varepsilon 0, \Omega) \cap H_0(\mathbf{curl}, \Omega).
\end{aligned}
$$

Definitions ctd

$$
\begin{aligned}
J_\nu^\star(\Omega,\varepsilon,\mu) \quad &= \{\chi \in X_T^0(\Omega,\mu) |\, \mathbf{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\chi) \in L^2(\Omega)^3 \\
&\quad \text{and } \mathbf{curl}\,\chi \times \nu = 0 \text{ on } \Gamma\},\\
J_T^\star(\Omega,\varepsilon,\mu) \quad &= \{\chi \in X_N^0(\Omega,\varepsilon) |\, \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\chi) \in L^2(\Omega)^3 \\
&\quad \text{and } \mathbf{curl}\,\chi \cdot \nu = 0 \text{ on } \Gamma\}.
\end{aligned}
$$

Some properties

Lemma

The space $H(\mathrm{div}\varepsilon 0, \Omega)$ *is equal to the closure in* $L^2(\Omega)^3$ *of*

$$X = \{\varphi \in L^2(\Omega)^3 | \varepsilon\varphi \in C^\infty(\bar{\Omega}) \text{ and } \mathrm{div}(\varepsilon\varphi) = 0\}.$$

Similarly, the space $H_0(\mathrm{div}\varepsilon 0, \Omega)$ *is equal to the closure in* $L^2(\Omega)^3$ *of*

$$\hat{X} = \{\varphi \in L^2(\Omega)^3 | \varepsilon\varphi \in \mathcal{D}(\Omega) \text{ and } \mathrm{div}(\varepsilon\varphi) = 0\}.$$

In [Lagnese 89], the spaces $H(\mathrm{div}0, \Omega)$ and $H_0(\mathrm{div}0, \Omega)$ are defined as in this Lemma.

Pf

Let us first assume that $\varepsilon = 1$. For the first one, let us fix $u \in H(\mathrm{div}0, \Omega)$. Then by Theorem I.3.4 of [Girault-Raviart, 86], there exists $\psi_0 \in H^1(\Omega)^3$ such that

$$u = \mathbf{curl}\, \psi_0.$$

Since $C^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$, there exists a sequence of $\psi_n \in C^\infty(\bar{\Omega})^3$ such that

$$\mathbf{curl}\, \psi_n \to \mathbf{curl}\, \psi_0 = u \text{ in } L^2(\Omega)^3, \text{ as } n \to \infty.$$

But $\mathbf{curl}\, \psi_n \in C^\infty(\bar{\Omega})$ and is divergence-free, hence the conclusion.
For an arbitrary ε, we simply use the equivalence

$$\varphi \in H(\mathrm{div}\varepsilon 0, \Omega) \Leftrightarrow \varepsilon\varphi \in H(\mathrm{div}0, \Omega).$$

The second assertion is proved similarly with $\psi_0 \in H_0^1(\Omega)^3$, see Theorem 3.20 of [Amrouche-Bernardi-Dauge-Girault, 98], and using the density of $\mathcal{D}(\Omega)$ into $H_0^1(\Omega)$.

Theorem

There exist two positive constants c_1, c_2 such that

$$\|\chi\|_{X_T^0(\Omega,\mu)} \leq c_1 \|\,\mathbf{curl}\,\chi\|_{L^2(\Omega)^3}, \forall \chi \in X_T^0(\Omega,\mu), \qquad (3)$$

$$\|\chi\|_{X_N^0(\Omega,\varepsilon)} \leq c_2 \|\,\mathbf{curl}\,\chi\|_{L^2(\Omega)^3}, \forall \chi \in X_N^0(\Omega,\varepsilon). \qquad (4)$$

Proof: Based on the compact embeddings of $X_T^0(\Omega,\mu)$ and $X_N^0(\Omega,\varepsilon)$ into $L^2(\Omega)^3$ [Weber, 80]. ■

For further purposes, we need the orthogonal projection P_ε on $H(\mathrm{div}\varepsilon 0, \Omega)$ in $L^2(\Omega)^3$, endowed with the inner product

$$(\chi, \varphi)_\varepsilon = \int_\Omega \varepsilon\chi \cdot \varphi \, dx.$$

Lemma (Le 3)

For any $\chi \in \mathbf{C}^\infty(\bar{\Omega})^3$, $\mathbf{curl}(P_\varepsilon \chi)$ *belongs to* $H(\mathbf{curl}, \Omega)$ *and satisfies*

$$\mathbf{curl}(P_\varepsilon \chi) = \mathbf{curl}\, \chi \text{ in } \Omega,$$
$$P_\varepsilon \chi \times \nu = \chi \times \nu \text{ on } \Gamma.$$

Pf

First take $\varphi \in \mathcal{D}(\Omega)^3$, then

$$\int_\Omega \mathbf{curl}(P_\varepsilon \chi) \cdot \varphi = \int_\Omega (P_\varepsilon \chi) \cdot \mathbf{curl}\, \varphi = \int_\Omega \varepsilon (P_\varepsilon \chi) \cdot \varepsilon^{-1} \mathbf{curl}\, \varphi.$$

As $\varepsilon^{-1} \mathbf{curl}\, \varphi \in H(\mathrm{div}\varepsilon 0, \Omega)$, we obtain

$$\int_\Omega \mathbf{curl}(P_\varepsilon \chi) \cdot \varphi = \int_\Omega \varepsilon \chi \cdot \varepsilon^{-1} \mathbf{curl}\, \varphi = \int_\Omega \mathbf{curl}\, \chi \cdot \varphi.$$

This proves the first identity.

Second take $\varphi \in H^1(\Omega)^3$, then by Green's formula

$$\int_\Omega \mathbf{curl}(P_\varepsilon\chi)\cdot\varphi = \int_\Omega (P_\varepsilon\chi)\cdot\mathbf{curl}\,\varphi + \langle (P_\varepsilon\chi)\times\nu;\varphi\rangle.$$

Again as $\varepsilon^{-1}\,\mathbf{curl}\,\varphi \in H(\mathrm{div}\varepsilon 0,\Omega)$, we obtain

$$\int_\Omega \mathbf{curl}(P_\varepsilon\chi)\cdot\varphi = \int_\Omega \varepsilon\chi\cdot\varepsilon^{-1}\,\mathbf{curl}\,\varphi + \langle (P_\varepsilon\chi)\times\nu;\varphi\rangle.$$

Again Green's formula gives

$$\int_\Omega \mathbf{curl}(P_\varepsilon\chi)\cdot\varphi = \int_\Omega \mathbf{curl}\,\chi\cdot\varphi - \langle \chi\times\nu;\varphi\rangle + \langle (P_\varepsilon\chi)\times\nu;\varphi\rangle.$$

By the first identity we obtain

$$\langle (P_\varepsilon\chi)\times\nu - \chi\times\nu;\varphi\rangle = 0,$$

and the second identity follows.

Corollary

The space $X_N^0(\Omega, \varepsilon)$ is dense in $H(\mathrm{div}\varepsilon 0, \Omega)$, while $X_T^0(\Omega, \mu)$ is dense in $H_0(\mathrm{div}\mu 0, \Omega)$.

Proof: As $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, the subspace $P_\varepsilon \mathcal{D}(\Omega)^3$ is clearly dense in $H(\mathrm{div}\varepsilon 0, \Omega)$. The first density result is proved since the inclusion

$$P_\varepsilon \mathcal{D}(\Omega)^3 \subset X_N^0(\Omega, \varepsilon)$$

follows from the previous Lemma.

The second density result is similarly proved by considering the orthogonal projection on $H_0(\mathrm{div}\mu 0, \Omega)$ wrt the inner product $(\cdot, \cdot)_\mu$.

A Green formula

Lemma

For all $\varphi \in H_0(\mathbf{curl}, \Omega)$ *and* $\psi \in H(\mathbf{curl}, \Omega)$, *we have*

$$\int_\Omega \mathbf{curl}\, \psi \cdot \varphi \, dx = \int_\Omega \mathbf{curl}\, \varphi \cdot \psi \, dx.$$

Proof: By section I.2.3 of [Girault-Raviart 86], $\mathcal{D}(\Omega)^3$ is dense in $H_0(\mathbf{curl}, \Omega)$, hence $\exists \varphi_n \in \mathcal{D}(\Omega)^3$ s.t.

$$\varphi_n \to \varphi \text{ in } H_0(\mathbf{curl}, \Omega).$$

Standard Green's formula $\Rightarrow$

$$\int_\Omega \mathbf{curl}\, \psi \cdot \varphi_n \, dx = \int_\Omega \mathbf{curl}\, \varphi_n \cdot \psi \, dx.$$

Taking the limit on n, we arrive at the conclusion.

Outline of the talk

1. The conservative systems
   - Two systems
   - Assumptions
2. Functions spaces
3. Well-posedness
4. The dissipative system
5. Density results and consequences
6. Well-posedness

The adjoint Maxwell equation

The homogeneous adjoint problem to (1) is

$$\left\{\begin{array}{l} \mu\frac{\partial\varphi}{\partial t} - \textbf{curl}\,\psi = 0 \text{ in } Q, \\ \varepsilon\frac{\partial\psi}{\partial t} + \textbf{curl}\,\varphi = 0 \text{ in } Q, \\ \mathrm{div}(\mu\varphi) = \mathrm{div}(\varepsilon\psi) = 0 \text{ in } Q, \\ \varphi\times\nu = 0, \psi\cdot\nu = 0 \text{ on } \Sigma, \\ \varphi(0) = \varphi_0,\ \psi(0) = \psi_0 \text{ in } \Omega. \end{array}\right. \tag{5}$$

First order system

Introduce the Hilbert space

$$H = H(\mathrm{div}\mu 0, \Omega) \times H_0(\mathrm{div}\varepsilon 0, \Omega),$$

equipped with the inner product

$$\left(\begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} \right)_H = \int_\Omega \{\mu\varphi\bar{\varphi}_1 + \varepsilon\psi\bar{\psi}_1\}\, dx.$$

Define the operator A as

$$D(A) = X_N^0(\Omega, \mu) \times X_T^0(\Omega, \varepsilon),$$

$$A \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \mu^{-1}\,\mathbf{curl}\,\psi \\ -\varepsilon^{-1}\,\mathbf{curl}\,\varphi \end{pmatrix}.$$

Formally problem (5) is equivalent to

$$\left\{ \begin{array}{l} \frac{\partial \Phi}{\partial t} = A\Phi, \\ \Phi(0) = \Phi_0, \end{array} \right. \tag{6}$$

when $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ and $\Phi_0 = \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}$.

We shall prove that this problem (6) has a unique solution using Lumer-Phillips' theorem:

Theorem

A linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ *generates a continuous semi-group* $\{S(t)\}_{t\geq 0}$ *of contractions on* H *if and only if*

1. $\Re(\mathcal{A}u, u)_H \leq 0, \forall u \in D(\mathcal{A})$,
2. $\exists \lambda > 0, \lambda I - \mathcal{A}$ *is surjective.*

Recall that a family of continuous linear operators $\{S(t)\}_{t\geq 0}$ is called a continuous semi-group of contractions if and only if

1. $S(0) = Id$,
2. $S(t)S(s) = S(t+s), \forall s, t \geq 0$,
3. $t \to S(t)x$ is continuous from $[0, \infty) \to H$ for all $x \in H$,
4. $\|S(t)\|_{\mathcal{L}(H)} \leq 1, \forall t \geq 0$.

The main properties of $\{S(t)\}_{t\geq 0}$ are that for all $x \in D(\mathcal{A})$, the mapping

$$t \to S(t)x,$$

is differentiable in $[0, \infty)$ and

$$\frac{d}{dt}S(t)x = \mathcal{A}S(t)x.$$

Hence u defined by

$$u(t) = S(t)x, \forall t \geq 0,$$

satisfies

$$\left\{ \begin{array}{l} \frac{du}{dt} = \mathcal{A}u \text{ in } H, \\ u(0) = x. \end{array} \right. \tag{7}$$

Maximality

Lemma

A and $-A$ are maximal dissipative operators.

Proof: 1. Dissipativeness of $\pm A$:

$$\Re(A\Phi, \Phi)_H = 0, \forall \Phi \in D(A). \tag{8}$$

With the above notation we have

$$(A\Phi, \Phi)_H = \int_\Omega \{\mathbf{curl}\, \psi \cdot \bar{\varphi} - \mathbf{curl}\, \varphi \cdot \bar{\psi}\}\, dx.$$

Green's formula (since $\psi, \varphi \in H_0(\mathbf{curl}, \Omega)) \Rightarrow$

$$\int_\Omega \mathbf{curl}\, \psi \cdot \bar{\varphi}\, dx = \overline{\int_\Omega \mathbf{curl}\, \varphi \cdot \bar{\psi}\, dx}.$$

The real part of this identity yields (8).

2. Maximality: This means that for all $(f,g)^\top$ in H, we are looking for $(\varphi,\psi)^\top$ in $D(A)$ such that

$$(I \pm A)\,(\varphi,\psi)^\top = (f,g)^\top .$$

Equivalently, we have

$$\psi = g \pm \varepsilon^{-1}\,\mathbf{curl}\,\varphi,$$

and

$$\varphi + \mu^{-1}\,\mathbf{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\varphi) = f \mp \mu^{-1}\,\mathbf{curl}\,g.$$

This last problem has a unique sol. φ in $X_N^0(\Omega,\mu)$ because its variational formulation is

$$\int_\Omega \{\varepsilon^{-1}\,\mathbf{curl}\,\varphi\,\mathbf{curl}\,\bar\theta + \mu\varphi\bar\theta\}\,dx = \int_\Omega \{\mu f\bar\theta \mp g\,\mathbf{curl}\,\bar\theta\}\,dx, \forall\theta \in X_N^0(\Omega,\mu).$$

Existence and uniqueness by Lax-Milgram lemma.

Lumer-Phillips' theorem $\Rightarrow A$ generates a C_0-group of contractions $T(t)$. Therefore we have the following existence result.

Theorem

For all $\Phi_0 \in H$, the problem (6) has a weak solution $\Phi \in C([0,\infty), H)$ given by $\Phi = T(t)\Phi_0$.
If moreover $\Phi_0 \in D(A^k)$, with $k \in \mathbf{N}^*$, the problem (6) has a strong solution $\Phi \in C([0,\infty), D(A^k)) \cap C^1([0,\infty), D(A^{k-1}))$.

Remark

If $k = 1$ or 2, we recover the results from [Lagnese 89] because

$$D(A^2) = J^\star_\tau(\Omega,\mu,\varepsilon) \times J^\star_\nu(\Omega,\mu,\varepsilon).$$

Energies

Lemma

If $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ *is a weak solution of problem (6) (or equivalently (5)), define the energy at time t by*

$$E(t) = \frac{1}{2} \int_\Omega \{\mu |\varphi(x,t)|^2 + \varepsilon |\psi(x,t)|^2\}\, dx.$$

Then we have

$$E(t) = E(0), \forall t \geq 0. \tag{9}$$

Proof: Since $D(A)$ is dense in H it suffices to prove (9) for $\begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}$ in $D(A)$. For such a initial datum, φ and ψ are differentiable and therefore

$$\begin{aligned} \frac{d}{dt}E(t) &= \Re \int_\Omega \{\mu \frac{\partial \varphi}{\partial t}\bar{\varphi} + \varepsilon \frac{\partial \psi}{\partial t}\bar{\psi}\}\, dx \\ &= \Re \int_\Omega \{\mathbf{curl}\, \psi \bar{\varphi} - \mathbf{curl}\, \varphi \bar{\psi}\}\, dx \\ &= \Re(A \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix})_H = 0, \end{aligned}$$

due to (8). ∎

Lemma

If $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ *is a strong solution of problem (6) (or equivalently (5)) with initial datum in* $D(A)$*, define the modified energy at time* t *by*

$$\tilde{E}(t) = \frac{1}{2}\int_\Omega \{\mu^{-1}|\,\mathbf{curl}\,\psi(x,t)|^2 + \varepsilon^{-1}|\,\mathbf{curl}\,\varphi(x,t)|^2\}\,dx.$$

Then we have

$$\tilde{E}(t) = \tilde{E}(0), \forall t \geq 0. \tag{10}$$

Proof: Since $D(A^2)$ is dense in $D(A)$ it suffices to prove (10) for $\begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}$ in $D(A^2)$.

For such a initial datum, φ and ψ are C^2 in time, and $(\varphi_t, \psi_t)^\top \in D(A)$. Furthermore

$$\tilde{E}(t) = \frac{1}{2} \int_\Omega \{\mu|\varphi_t(x,t)|^2 + \varepsilon|\psi_t(x,t)|^2\}\, dx.$$

Hence the same calculations as before yield

$$\frac{d}{dt}\tilde{E}(t) = \Re(A \begin{pmatrix} \varphi_t \\ \psi_t \end{pmatrix}, \begin{pmatrix} \varphi_t \\ \psi_t \end{pmatrix})_H = 0,$$

due to (8). ∎

Outline of the talk

Silver-Müller bc

We consider (non-stationary) Maxwell's equations:

$$\left\{\begin{array}{l} \varepsilon\frac{\partial E}{\partial t} - \mathbf{curl}\, H = 0 \text{ in } Q = \Omega\times]0,+\infty[, \\ \mu\frac{\partial H}{\partial t} + \mathbf{curl}\, E = 0 \text{ in } Q, \\ \mathrm{div}(\varepsilon E) = \mathrm{div}(\mu H) = 0 \text{ in } Q, \\ H\times\nu + (E\times\nu)\times\nu = 0 \text{ on } \Sigma := \Gamma\times]0,+\infty[, \\ E(0) = E_0,\ H(0) = H_0 \text{ in } \Omega, \end{array}\right. \tag{11}$$

where ν denotes the unit outer normal vector on Γ. This means that we suppose that the time evolution of the electric field E and the magnetic field H is driven by a damping on Γ.
The boundary condition is the so-called Silver-Müller boundary condition.

First order system

Introduce the Hilbert space

$$\mathcal{H} = H(\mathrm{div}\varepsilon 0, \Omega) \times H(\mathrm{div}\mu 0, \Omega),$$

equipped with the inner product

$$\left((\varphi, \psi)^\top, (\varphi_1, \psi_1)^\top\right)_{\mathcal{H}} = \int_\Omega \{\varepsilon\varphi\bar{\varphi}_1 + \mu\psi\bar{\psi}_1\}\, dx.$$

Define the operator A as

$$\begin{aligned} D(A) = \{&(E,H)^\top \in \mathcal{H} |\, \mathbf{curl}\, E, \mathbf{curl}\, H \in L^2(\Omega)^3; \\ &E \times \nu, H \times \nu \in L^2(\Gamma)^3 \text{ satisfying} \\ &H \times \nu + (E \times \nu) \times \nu = 0 \text{ on } \Gamma\}, \end{aligned} \tag{12}$$

$$A(E,H)^\top = \left(\varepsilon^{-1}\,\mathbf{curl}\, H, -\mu^{-1}\,\mathbf{curl}\, E\right)^\top, \forall (E,H)^\top \in D(A).$$

Formally problem (11) is equivalent to

$$\left\{\begin{array}{l} \frac{\partial \Phi}{\partial t} = A\Phi, \\ \Phi(0) = \Phi_0, \end{array}\right. \tag{13}$$

when $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ and $\Phi_0 = \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}$.

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A density result

Recall the next density result from [Ben Belgacem-Bernardi-Costabel-Dauge, 97]

Theorem

Introduce the Hilbert space

$$W = \{E \in L^2(\Omega)^3 | \,\mathbf{curl}\, E \in L^2(\Omega)^3 \text{ and } E \times \nu \in L^2(\Gamma)^3\}, \quad (14)$$

with the norm

$$||E||_W^2 = \int_\Omega (|E|^2 + |\,\mathbf{curl}\, E|^2) dx + \int_\Gamma |E \times \nu|^2 d\sigma.$$

Then $H^1(\Omega)^3$ is dense in W.

An ibp formula

Lemma

For all $\begin{pmatrix} E \\ H \end{pmatrix} \in D(A)$, *one has*

$$\int_\Omega (\mathbf{curl}\, E \cdot H - \mathbf{curl}\, H \cdot E)\, dx = \int_\Gamma H \times \nu \cdot E d\sigma. \tag{15}$$

Proof: We first remark that (15) holds for all $\begin{pmatrix} E \\ H \end{pmatrix}$ in $H^1(\Omega)^3 \times H^1(\Omega)^3$ owing to Green's formula. By density (see the previous Theorem) it still holds in $W \times W$. We conclude since $D(A)$ is clearly continuously embedded into $W \times W$.

Another density result

We next prove the following density result, which is closely related to Lemma 1.

Lemma

The domain of the operator A is dense in $\mathcal{H}$.

Proof: Recall that P_ε is the projection on $H(\mathrm{div}\varepsilon 0, \Omega)$ in $L^2(\Omega)^3$ endowed with the inner product $(\cdot, \cdot)_\varepsilon$. As $\mathcal{D}(\Omega)^3$ is dense in $L^2(\Omega)^3$, $P_\varepsilon \mathcal{D}(\Omega)^3$ is dense in $H(\mathrm{div}\varepsilon 0, \Omega)$. Consequently $P_\varepsilon \mathcal{D}(\Omega)^3 \times P_\mu \mathcal{D}(\Omega)^3$ is dense in $\mathcal{H} = H(\mathrm{div}\varepsilon 0, \Omega) \times H(\mathrm{div}\mu 0, \Omega)$.

Moreover by Lemma 1 for any $\chi \in \mathcal{D}(\Omega)^3$, we have

$$\mathbf{curl}(P_\varepsilon \chi) = \mathbf{curl}\, \chi \text{ in } \Omega,$$
$$P_\varepsilon \chi \times \nu = 0 \text{ on } \Gamma.$$

We then conclude that

$$P_\varepsilon \mathcal{D}(\Omega)^3 \times P_\mu \mathcal{D}(\Omega)^3 \subset D(A).$$

Therefore the above density result implies that $D(A)$ is dense in $\mathcal{H}$.

Outline of the talk

Dissipativeness

Lemma

A is a dissipative operator.

Proof: Dissipativeness of $A \Leftrightarrow$

$$\Re(A\Phi, \Phi)_{\mathcal{H}} \leq 0, \forall \Phi \in D(A).$$

With the notation $\Phi = (E, H)^{\top}$, we have
$(A\Phi, \Phi)_{\mathcal{H}} = \int_{\Omega}\{\mathbf{curl}\, H\bar{E} - \mathbf{curl}\, E\bar{H}\}\, dx.$
Green's formula (see (15)) $\Rightarrow$
$(A\Phi, \Phi)_{\mathcal{H}} = \int_{\Gamma} \bar{H} \times \nu \cdot E d\sigma.$
Using the boundary condition (12), we arrive at

$$(A\Phi, \Phi)_{\mathcal{H}} = -\int_{\Gamma} |E \times \nu|^2 d\sigma.$$

Maximality

Lemma

A is maximal .

Corollary

The domain of the operator A is dense in $\mathcal{H}$.

Pf

For all $\left(\begin{array}{c} F \\ G \end{array}\right)$ in $\mathcal{H}$, we are looking for $\left(\begin{array}{c} E \\ H \end{array}\right)$ in $D(A)$ such that

$$(I+A)\left(\begin{array}{c} E \\ H \end{array}\right)=\left(\begin{array}{c} F \\ G \end{array}\right). \tag{16}$$

Formally, we then have

$$H = G - \mu^{-1} \,\mathbf{curl}\, E, \tag{17}$$

and

$$\varepsilon E + \mathbf{curl}(\mu^{-1} \,\mathbf{curl}\, E) = \varepsilon F + \mathbf{curl}\, G. \tag{18}$$

This last equation in E will have a unique solution by adding a boundary condition on E. Indeed using the identity (17), we see that (12) is formally equivalent to

$$-\mu^{-1} \,\mathbf{curl}\, E \times \nu + (E \times \nu) \times \nu = -G \times \nu \text{ on } \Gamma. \tag{19}$$

The variational formulation of problem (18)-(19): Find $E \in W_\varepsilon$ such that

$$a(E, E') = \int_\Omega \{\varepsilon F \cdot E' + G \cdot \textbf{curl}\, E'\}\, dx, \forall E' \in W_\varepsilon, \tag{20}$$

where the Hilbert space W_ε is defined by

$$W_\varepsilon = \{E \in L^2(\Omega)^3 |\, \textbf{curl}\, E \in L^2(\Omega)^3, \mathrm{div}(\varepsilon E) \in L^2(\Omega) \text{ and } E \times \nu \in L^2(\Gamma)^3\},$$

with the norm

$$||E||^2_{W_\varepsilon} = \int_\Omega (|E|^2 + |\,\textbf{curl}\, E|^2 + |\,\mathrm{div}(\varepsilon E)|^2) dx + \int_\Gamma |E \times \nu|^2 d\sigma.$$

and the form a is defined by

$$\begin{aligned} a(E, E') \quad &= \quad \int_\Omega \{\mu^{-1}\, \textbf{curl}\, E \cdot \textbf{curl}\, E' + \varepsilon E \cdot E' + s\, \mathrm{div}(\varepsilon E)\, \mathrm{div}(\varepsilon E')\}\, dx \\ &+ \quad \int_\Gamma (E \times \nu) \cdot E' \times \nu\, d\sigma, \end{aligned}$$

$s > 0$ being a parameter appropriately chosen later on.

As

$$a(E,E) = \int_\Omega \{\mu^{-1}|\,\mathbf{curl}\,E|^2 + \varepsilon|E|^2 + s|\operatorname{div}(\varepsilon E)|^2\}\,dx + \int_{\Gamma^+} |E\times\nu|^2\,d\sigma,$$

the sesquilinear form a is coercive on W_ε and by Lax-Milgram lemma, there exists a unique sol. $E \in W_\varepsilon$ of (20).

At this stage we need to show that this solution $E \in W_\varepsilon$ of (20) and H given by (17) are such that the pair $\begin{pmatrix} E \\ H \end{pmatrix}$ belongs to $D(A)$ and satisfies (16).

We first show that εE is divergence free: Take test functions $E' = \nabla\phi$ with $\phi \in D(\Delta_\varepsilon^{Dir})$, where $D(\Delta_\varepsilon^{Dir})$ is the domain of the operator Δ_ε^{Dir} with Dirichlet boundary conditions defined by

$$D(\Delta_\varepsilon^{Dir}) = \{\phi \in H_0^1(\Omega) | \Delta_\varepsilon \phi := \operatorname{div}(\varepsilon \nabla \phi) \in L^2(\Omega)\},$$
$$\Delta_\varepsilon^{Dir}\phi = \Delta_\varepsilon \phi, \forall \phi \in D(\Delta_\varepsilon^{Dir}).$$

In that case (20) becomes (the boundary term disappears since ϕ is zero on Γ)

$$\int_\Omega \{\varepsilon E \cdot \nabla\phi + s \operatorname{div}(\varepsilon E)\Delta_\varepsilon \phi\}\, dx = \int_\Omega \varepsilon F \cdot \nabla\phi\, dx, \forall \phi \in D(\Delta_\varepsilon^{Dir}).$$

Since εE and εF have a divergence in $L^2(\Omega)$, by Green's formula in the above left-hand side and right-hand side (allowed since ϕ is in $H^1(\Omega)$), we obtain

$$\int_\Omega \operatorname{div}(\varepsilon E)\{\phi + s\Delta_\varepsilon \phi\}\, dx = 0, \forall \phi \in D(\Delta_\varepsilon^{Dir}),$$

since εF is divergence free. Taking $s > 0$ such that $-s^{-1}$ is not an eigenvalue of Δ_ε^{Dir} (always possible since Δ_ε^{Dir} is a negative selfadjoint operator with a discrete spectrum), we conclude that

$$\operatorname{div}(\varepsilon E) = 0 \text{ in } \Omega.$$

Using this fact and the identity (17), we see that (20) is equivalent to

$$\int_\Omega \{\varepsilon E \cdot E' - H \cdot \mathbf{curl}\, E'\}\, dx + \int_\Gamma (E \times \nu) \cdot E' \times \nu \, d\sigma$$
$$= \int_\Omega \varepsilon F \cdot E' \, dx, \forall E' \in W_\varepsilon.$$

Taking first test functions $E' = P_\varepsilon \chi$ with $\chi \in \mathcal{D}(\Omega)^3$ by Lemma 16 we get

$$\varepsilon E - \mathbf{curl}\, H = \varepsilon F \text{ in } \mathcal{D}'(\Omega).$$

This means that the first identity in (16) holds since the above identity yields $\mathbf{curl}\, H \in L^2(\Omega)$.
Now taking test functions $E' = P_\varepsilon \chi$ with $\chi \in C^\infty(\bar\Omega)^3$ by Lemma 3 and Green's formula (see (15)), we get the BC:

$$H \times \nu + (E \times \nu) \times \nu = 0 \text{ on } \Gamma.$$

Finally from (17) and the fact that μG is divergence free, μH is also divergence free.

Lumer-Phillips' theorem $\Rightarrow A$ generates a C_0-group of contractions $(T(t))_{t\geq 0}$. Therefore we have the following existence result.

Theorem

For all $\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \in \mathcal{H}$, *the problem (11) admits a unique (weak) solution* $\begin{pmatrix} E \\ H \end{pmatrix} \in C(\mathbf{R}_+, \mathcal{H})$. *If moreover* $\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \in D(A)$, *the problem (11) admits a unique (strong) solution* $\begin{pmatrix} E \\ H \end{pmatrix} \in C^1(\mathbf{R}_+, \mathcal{H}) \cap C(\mathbf{R}_+, D(A))$.