Stability and controllability results for the heterogeneous Maxwell equations

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September 20, 2012

Outline of the talk

The problem

We consider (non-stationary) Maxwell's equations:

$$\left\{\begin{array}{l} \varepsilon \frac{\partial E}{\partial t} - \mathbf{curl}\, H = 0 \text{ in } Q = \Omega \times]0, +\infty[, \\ \mu \frac{\partial H}{\partial t} + \mathbf{curl}\, E = 0 \text{ in } Q, \\ \operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0 \text{ in } Q, \\ H \times \nu + (E \times \nu) \times \nu = 0 \text{ on } \Sigma := \Gamma \times]0, +\infty[, \\ E(0) = E_0,\ H(0) = H_0 \text{ in } \Omega, \end{array}\right. \tag{1}$$

where ν denotes the unit outer normal vector on Γ. This means that we suppose that the time evolution of the electric field E and the magnetic field H is driven by a damping on Γ.
The boundary condition is the so-called Silver-Müller boundary condition.

Goal:

Find sufficient conditions that garantee that the energy

$$\mathcal{E}(t) = \frac{1}{2}\int_{\Omega}(\varepsilon|E(x,t)|^2 + \mu|H(x,t)|^2)\,dx$$

of the system decays exponentially.

Motivation

There exists a vast literature concerning stability of hyperbolic PDE. Let us quote

- Wave equation: Komornik 91, 94, Komornik-Zuazua 90, Lasiecka-Triggiani 92
- Petrovsky system: Guesmia 98, 99
- Elastodynamic system: Alalbau-Komronik 99, Guesmia 99, Bey-Hemina-Lohéac 03
- Maxwell system: Komornik 94, Phung 00, N.-Eller-Lagnese 02

Similar structure: Use observability estimates+ invariance by a time translation of the problem.

General principle

1. observability estimates $\Leftrightarrow$ exp. stab.
2. Checking the OE.
3. Russell's ppl: Exp. stab $\Rightarrow$ EC results.

GOAL: Explain this scheme for heterogeneous Maxwell system.

Rk An abstract setting is also possible.

Assumptions on the domain and on the coefficients

Ω is a bounded, simply connected domain with a Lipschitz boundary Γ.

ε and μ are piecewise constant on Lipschitz polyhedral subdomains, in the sense that we assume that there exists a partition $\mathcal{P}$ of Ω in a finite set of Lipschitz polyhedra $\Omega_1, \cdots, \Omega_J$ such that on each Ω_j, $\varepsilon = \varepsilon_j$ and $\mu = \mu_j$, where ε_j and μ_j are positive constants.

A Lipschitz polyhedron is a bounded, simply connected Lipschitz domain with piecewise plane boundary.

Steps

- Introduce and analyze adapted function spaces,
- Existence results follow from semi-group theory,
- Exponential stability of the system using an observability estimate,
- Exact controllability problem via Russell's principle,

Outline of the talk

Definitions

$$
\begin{aligned}
H(\mathrm{div}\varepsilon 0, \Omega) &= \{\chi \in L^2(\Omega)^3 | \mathrm{div}(\varepsilon\chi) = 0\}, \\
H_0(\mathrm{div}\varepsilon 0, \Omega) &= \{\chi \in H(\mathrm{div}\varepsilon 0, \Omega) | \chi \cdot \nu = 0 \text{ on } \Gamma\}, \\
H(\mathbf{curl}, \Omega) &= \{\chi \in L^2(\Omega)^3 | \mathbf{curl}\, \chi \in L^2(\Omega)^3\}, \\
H_0(\mathbf{curl}, \Omega) &= \{\chi \in H(\mathbf{curl}, \Omega) | \chi \times \nu = 0 \text{ on } \Gamma\}, \\
X_T^0(\Omega, \mu) &= H_0(\mathrm{div}\mu 0, \Omega) \cap H(\mathbf{curl}, \Omega), \\
X_N^0(\Omega, \varepsilon) &= H(\mathrm{div}\varepsilon 0, \Omega) \cap H_0(\mathbf{curl}, \Omega).
\end{aligned}
$$

First order system

Introduce the Hilbert space

$$\mathcal{H} = H(\mathrm{div}\varepsilon 0, \Omega) \times H(\mathrm{div}\mu 0, \Omega),$$

equipped with the inner product

$$\left((\varphi, \psi)^\top, (\varphi_1, \psi_1)^\top\right)_{\mathcal{H}} = \int_\Omega \{\mu\varphi\bar{\varphi}_1 + \varepsilon\psi\bar{\psi}_1\}\, dx.$$

Define the operator A as

$$\begin{aligned} D(A) = \{&(E, H)^\top \in \mathcal{H} |\, \mathbf{curl}\, E, \mathbf{curl}\, H \in L^2(\Omega)^3; \\ &E \times \nu, H \times \nu \in L^2(\Gamma)^3 \text{ satisfying} \\ &H \times \nu + (E \times \nu) \times \nu = 0 \text{ on } \Gamma\}, \\ A\,(E, H)^\top &= \left(\varepsilon^{-1}\,\mathbf{curl}\, H, -\mu^{-1}\,\mathbf{curl}\, E\right)^\top, \forall\, (E, H)^\top \in D(A). \end{aligned}$$

Formally problem (1) is equivalent to

$$\left\{ \begin{array}{l} \frac{\partial \Phi}{\partial t} = A\Phi, \\ \Phi(0) = \Phi_0, \end{array} \right. \tag{2}$$

when $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ and $\Phi_0 = \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}$.

We have proved that this problem (2) has a unique solution using semigroup theory.

Lemma

A is a maximal dissipative operator.

Corollary

The domain of the operator A is dense in $\mathcal{H}$.

Semigroup theory allows to conclude the following existence results:

Theorem

For all $\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \in \mathcal{H}$, *the problem (1) admits a unique (weak) solution* $\begin{pmatrix} E \\ H \end{pmatrix} \in C(\mathbf{R}_+, \mathcal{H})$. *If moreover* $\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \in D(A)$, *the problem (1) admits a unique (strong) solution* $\begin{pmatrix} E \\ H \end{pmatrix} \in C^1(\mathbf{R}_+, \mathcal{H}) \cap C(\mathbf{R}_+, D(A))$.

Energy

Lemma

If $\Phi = (E, H)^\top$ *is a weak solution of problem (2), then the energy*

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \{\varepsilon|E(t,x)|^2 + \mu|H(t,x)|^2\}\, dx \tag{3}$$

is non-increasing and

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \int_\Gamma |E(t) \times \nu|^2\, d\sigma dt, \forall 0 \leq S < T < \infty. \tag{4}$$

Pf

Since $D(A)$ is dense in $\mathcal{H}$ it suffices to show that

$$\mathcal{E}'(t) = -\int_\Gamma |E(t) \times \nu|^2 \, d\sigma. \tag{5}$$

for strong solutions (i.e. with initial data in $D(A)$). But for such a solution, we have

$$\mathcal{E}'(t) = \int_\Omega \{\varepsilon E(t,x) \cdot E'(t,x) + \mu H(t,x) \cdot H'(t,x)\} \, dx.$$

By (1) and Green's formula (+bc), we get

$$\begin{aligned}
\mathcal{E}'(t) &= \int_\Omega \{E(t,x) \cdot \mathbf{curl}\, H(t,x) - H(t,x) \cdot \mathbf{curl}\, E(t,x)\} \, dx \\
&= -\int_\Gamma (H \times \nu) \cdot E \, d\sigma = -\int_\Gamma |E(t) \times \nu|^2 \, d\sigma \leq 0.
\end{aligned}$$

Outline of the talk

A stability estimate

GOAL: Find necessary and sufficient conditions which guarantee the exponential stability of the energy of (1).

Definition

We say that the stability estimate holds if there exist $T > 0$ and two non negative constants C_1, C_2 (which may depend on T) with $C_1 < T$ such that

$$\int_0^T \mathcal{E}(t)\, dt \leq C_1 \mathcal{E}(0) + C_2 \int_0^T \int_\Gamma |E(t) \times \nu|^2 \, d\sigma dt, \qquad (6)$$

for all solution (E, H) of (1).

Stab. estimate: Equiv. formulation

Lemma (Le 6)

The stability estimate holds if and only if there exist $T > 0$ and a positive constant C (which may depend on T) such that

$$\mathcal{E}(T) \leq C \int_0^T \int_\Gamma |E(t) \times \nu|^2 \, d\sigma \, dt, \tag{7}$$

for all solution (E, H) of (1).

Remark

The estimate (7) is called an observabilty estimate since it allows to estimate the energy at time T by the observation of $E(t) \times \nu$ on Γ from 0 to T.

Proof

$\Rightarrow$:

$$
\begin{aligned}
T\mathcal{E}(T) &\leq \int_0^T \mathcal{E}(t)\,dt \text{ (energy decay)} \\
&\leq C_1\mathcal{E}(0) + C_2 \int_0^T \int_\Gamma |E(t) \times \nu|^2\, d\sigma\, dt \text{ (hyp. (6))} \\
&\leq C_1\mathcal{E}(T) + (C_1 + C_2) \int_0^T \int_\Gamma |E(t) \times \nu|^2\, d\sigma\, dt \text{ ((4))} .
\end{aligned}
$$

This yields (7) with $C = \frac{C_1+C_2}{T-C_1}$.

Proof ctd

$\Leftarrow$:

$$\int_0^T \mathcal{E}(t)\,dt \leq T\mathcal{E}(0) \text{ (energy decay)}$$
$$\leq \frac{T}{2}\mathcal{E}(0) + \frac{T}{2}(\mathcal{E}(T) + \int_0^T \int_\Gamma |E(t) \times \nu|^2\,d\sigma\,dt) \text{ ((4))}$$
$$\leq \frac{T}{2}\mathcal{E}(0)$$
$$+\frac{T}{2}(1+C)\int_0^T \int_\Gamma |E(t) \times \nu|^2\,d\sigma\,dt \text{ (hyp. (7))}$$

which is nothing else than (6).

Exponential stability

We now show that the stability estimate is equivalent to the exponential stability of (1).

Theorem

The stability estimate holds if and only if there exist two positive constants M and ω such that

$$\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0), \tag{8}$$

for all solution (E, H) of (1).

Proof: sufficiency

$\Rightarrow$ Assume that the stab. est. holds: By Le 6, (7) holds: $\Rightarrow$

$$\mathcal{E}(T) \leq C \int_0^T \int_\Gamma |E(t) \times \nu|^2 \, d\sigma \, dt.$$

The identity (4) then yields

$$\mathcal{E}(T) \leq C(\mathcal{E}(0) - \mathcal{E}(T)).$$

This estimate is equivalent to

$$\mathcal{E}(T) \leq \gamma \mathcal{E}(0),$$

with $\gamma = \frac{C}{1+C}$ which is < 1.

Pf: sufficiency ctd

Applying this argument on $[(m-1)T, mT]$, for $m = 1, 2, \cdots$ (valid since our system is invariant by a translation in time):

$$\mathcal{E}(mT) \leq \gamma \mathcal{E}((m-1)T) \leq \cdots \leq \gamma^m \mathcal{E}(0), m = 1, 2, \cdots$$

Therefore we have

$$\mathcal{E}(mT) \leq e^{-\omega m T} \mathcal{E}(0), m = 1, 2, \cdots$$

with $\omega = \frac{1}{T} \ln \frac{1}{\gamma} > 0$. For an arbitrary positive t, there exists $m = 1, 2, \cdots$ such that $(m-1)T < t \leq mT$ and by the nonincreasing property of $\mathcal{E}$, we conclude

$$\mathcal{E}(t) \leq \mathcal{E}((m-1)T) \leq e^{-\omega(m-1)T} \mathcal{E}(0) \leq \frac{1}{\gamma} e^{-\omega t} \mathcal{E}(0).$$

Pf: necessity

Assume exp stab: (4) $\Rightarrow$ for any $T > 0$

$$\int_0^T \int_\Gamma |E(t) \times \nu|^2 \, d\sigma \, dt = \mathcal{E}(0) - \mathcal{E}(T)$$
$$\geq \mathcal{E}(0)(1 - Me^{-\omega T}) \text{ thanks to (8).}$$

The exp. decay (8) also implies for all $C_1 > 0$

$$\int_0^T \mathcal{E}(t) dt \leq M\mathcal{E}(0) \frac{1 - e^{-\omega T}}{\omega}$$
$$\leq C_1 \mathcal{E}(0) + \left(\frac{M(1 - e^{-\omega T})}{\omega} - C_1 \right) \mathcal{E}(0).$$

Pf: necessity ctd

Choosing $T \gg$ s. t. $1 - Me^{-\omega T} > 0$ and $C_1 < \min\{\frac{M(1-e^{-\omega T})}{\omega}, T\}$, these two estimates yield (6) with

$$C_2 = \left(\frac{M(1-e^{-\omega T})}{\omega} - C_1\right)(1 - Me^{-\omega T})^{-1}.$$

Outline of the talk

The homogeneous case

If $\varepsilon = \mu = 1$, the SE (6) can be obtained by using microlocal analysis [Phung 00]:
Suppose that Ω is a connected domain with a smooth boundary Γ consisting of a single connected component and assume that $\varepsilon = \mu = 1$. Then the stability estimate holds.

The non homogeneous case

If ε and μ are non homogeneous, the SE (6) is usually obtained by using the multiplier method, see [Kapitonov 94, Martinez 99]:

Definition

We say that Ω is (ε, μ)-substarlike if $\exists \varphi \in W^{2,\infty}(\Omega)$ and $\alpha > 0$ satisfying

$$\Delta\varphi(x)|\xi|^2 - 2(D^2\varphi(x)\xi)\cdot\xi \geq \alpha|\xi|^2, \forall \xi \in \mathbf{R}^3, \forall \text{ a.e. } x \in \Omega, \tag{9}$$

$$\frac{\partial\varphi}{\partial\nu} > 0 \text{ on } \Gamma, \tag{10}$$

$$\frac{\partial\varphi}{\partial\nu_{i_F}}(\varepsilon_{i_F} - \varepsilon_{i'_F}) \leq 0, \text{ and } \frac{\partial\varphi}{\partial\nu_{i_F}}(\mu_{i_F} - \mu_{i'_F}) \leq 0 \text{ on } F, \forall F \in \mathcal{F}_{int}, \tag{11}$$

Hereabove and below, for any interior interface F (written in short $F \in \mathcal{F}_{int}$) the different indices i_F and i'_F are such that $F = \bar{\Omega}_{i_F} \cap \bar{\Omega}_{i'_F}$ and are fixed once and for all. For a fixed subdomain Ω_j, ν_j is the exterior unit normal vector along its boundary.

Some further spaces

Before giving some examples of domains being (ε, μ)-substarlike, let us show that this (geometrical) property and a density result (which could be interpreted as a geometrical property, cf. [Lohrengel-N. 01]) guarantee that Ω satisfies the stability estimate.
To formulate that result we recall that the space $PH^1(\Omega, \mathcal{P})$ of piecewise H^1 (scalar) function in Ω is defined by

$$PH^1(\Omega, \mathcal{P}) := \{u \in L^2(\Omega) | u_{|\Omega_j} \in H^1(\Omega_j), \forall j = 1, \cdots, J\}.$$

We further introduce

$$W_\varepsilon = \{E \in L^2(\Omega)^3 | \, \mathbf{curl}\, E \in L^2(\Omega)^3, \mathrm{div}(\varepsilon E) \in L^2(\Omega), E \times \nu \in L^2(\Gamma)^3\},$$

with the norm

$$||E||^2_{W_\varepsilon} = \int_\Omega (|E|^2 + |\, \mathbf{curl}\, E|^2 + |\, \mathrm{div}(\varepsilon E)|^2) dx + \int_\Gamma |E \times \nu|^2 d\sigma.$$

The multiplier method

Theorem

If Ω is (ε, μ)-substarlike and if $PH^1(\Omega, \mathcal{P})^3 \cap W_\varepsilon$ (resp. $PH^1(\Omega, \mathcal{P})^3 \cap W_\mu$) is dense in W_ε (resp. in W_μ), then the stability estimate

$$\int_0^T \mathcal{E}(t)\, dt \leq C_1 \mathcal{E}(0) + C_2 \int_0^T \int_\Gamma |E(t) \times \nu|^2 \, d\sigma dt$$

holds, for all solution (E, H) of (1).

Proof: It suffices to show that the estimate (6) holds for any strong solution $\begin{pmatrix} E \\ H \end{pmatrix}$ of (1) and appropriate constants T, C_1, C_2.

Pf ctd

Fixing a function $\varphi \in W^{2,\infty}(\Omega)$ from Definition 8 we define the multiplier $m = \nabla\varphi$. We prove that the following est. holds for all $t \geq 0$:

$$\mathcal{E}(t) \leq \frac{1}{\alpha} \int_\Omega \varepsilon\mu \frac{d}{dt} \{(E \times H) \cdot m\}\, dx + \frac{R_1}{2\alpha} I^0_{ext}, \tag{12}$$

where R_1 is a positive constant and we set

$$I^0_{ext} = \int_\Gamma (\mu|H \times \nu|^2 + \varepsilon|E \times \nu|^2)\, d\sigma \leq M_0 \int_\Gamma |E \times \nu|^2\, d\sigma.$$

Integrating the estimate (12) from 0 to T, we get

$$\int_0^T \mathcal{E}(t)\, dt \leq \frac{1}{\alpha} \int_\Omega \varepsilon\mu \left[(E(t) \times H(t)) \cdot m\right]_0^T\, dx + \frac{R_1 M_0}{2\alpha} \int_{\Sigma_T} |E \times \nu|^2\, d\sigma.$$

Pf ctd

Since we readily check that

$$\left| \int_\Omega \varepsilon\mu(E(t) \times H(t)) \cdot m \, dx \right| \leq C\mathcal{E}(t) \leq C\mathcal{E}(0),$$

we arrive at

$$\int_0^T \mathcal{E}(t) \, dt \leq \frac{2C}{\alpha}\mathcal{E}(0) + \frac{R_1 M_0}{2\alpha} \int_{\Sigma_T} |E \times \nu|^2 \, d\sigma.$$

This proves the stability estimate by choosing T large enough, i.e., $T > \frac{2C}{\alpha}$.

Pf of (12)

Starting from

$$I := \int_\Omega \varepsilon\mu \frac{d}{dt}\left\{(E \times H)\cdot m\right\}\, dx$$

and using the first two identities of (1) we get

$$I = \int_\Omega \left\{\mu(\mathbf{curl}\, H \times H)\cdot m + \varepsilon(\mathbf{curl}\, E \times E)\cdot m\right\}\, dx.$$

Assume for a moment that $(E, H)^\top$ belongs to
$(PH^1(\Omega,\mathcal{P})^3 \cap W_\varepsilon) \times (PH^1(\Omega,\mathcal{P})^3 \cap W_\mu)$. Using the standard identity

$$\mathbf{curl}\, H \times H = (H\cdot\nabla)H - \frac{1}{2}\nabla|H|^2,$$

we obtain

$$I = -\frac{1}{2}\int_\Omega \left\{\mu\nabla|H|^2\cdot m + \varepsilon\nabla|E|^2\cdot m\right\}\, dx + I_\mu(H) + I_\varepsilon(E),$$

where for shortness we have set

$$I_\mu(H) = \int_\Omega \mu\left((H\cdot\nabla)H\right)\cdot m\, dx.$$

Pf of (12) ctd

Using some Green's formulas we get

$$2I = \int_\Omega (\mu \mathcal{M} H \cdot H + \varepsilon \mathcal{M} E \cdot E)\, dx - 2I_0 - I_{\varepsilon,int}(E) - I_{\mu,int}(H) - I_{ext}, \qquad (13)$$

where we have set

$$\begin{aligned}
\mathcal{M} &= \operatorname{div} m I - 2Dm, \\
I_{\mu,int}(H) &= \sum_{F \in \mathcal{F}_{int}} \int_F \left\{ [\mu |H|^2]_F (m \cdot \nu_{i_F}) - 2[\mu (H \cdot \nu_{i_F})(H \cdot m)]_F \right\} d\sigma, \\
I_{ext} &= \int_\Gamma \left\{ m \cdot \nu (\varepsilon |E|^2 + \mu |H|^2) - 2\varepsilon (m \cdot E)(E \cdot \nu) - 2\mu (m \cdot H)(H \cdot \nu) \right\} d\sigma, \\
I_0 &= \int_\Omega (E \cdot m \operatorname{div}(\varepsilon E) + H \cdot m \operatorname{div}(\mu H))\, dx.
\end{aligned}$$

Assumptions (11) as well as the properties of E and H through the interior interfaces $\Rightarrow I_{\varepsilon,int}(E) \le 0, I_{\mu,int}(H) \le 0$. Assumption (9) $\Rightarrow \mathcal{M} X \cdot X \ge \alpha |X|^2$.

Combined with (13) we get

Pf of (12) ctd

$$2I \geq \alpha \int_{\Omega} (\mu |H|^2 + \varepsilon |E|^2)\, dx - 2I_0 - I_{ext}.$$

At this stage we remark that the assumption (10) guarantees that $m \cdot \nu > 0$ on Γ, therefore we can show that

$$I_{ext} \leq R_1 I^0_{ext},$$

Hence

$$2I \geq \alpha \int_{\Omega} (\mu |H|^2 + \varepsilon |E|^2)\, dx - 2I_0 - R_1 I^0_{ext}.$$

By the density assumptions this inequality remains valid in $W_\varepsilon \times W_\mu$ and, therefore, for any strong solution of (1) since $D(A)$ is continuously embedded into $W_\varepsilon \times W_\mu$. From the property $\mathrm{div}(\varepsilon E) = \mathrm{div}(\mu H) = 0$ it actually reduces to (12).

Second example: Nested domains

Assume that the domain Ω is strictly star-shaped with respect to the origin 0 and that the subdomains are nested in the following sense, see Figure **??** for a two-dimensional illustration: the origin belongs to Ω_1 and

$$\bar{\Omega}_j \cap \bar{\Omega}_{j+1} = \partial\Omega_j \setminus \partial\Omega_{j-1}, \forall j \geq 2.$$

Then $\varphi(x) = |x|^2/2$ directly satisfies (9) and (10), while (11) is equivalent to

$$\varepsilon_j \leq \varepsilon_{j+1} \text{ and } \mu_j \leq \mu_{j+1} \, \forall j = 1, \cdots, J.$$

Third example: The checker board

Assume that the domain Ω is strictly star-shaped with respect to the origin 0 and that the subdomains are around the origin in the sense that

$$x \cdot n = 0 \text{ on } \partial\Omega_j \setminus \Gamma.$$

This is equivalent to say that the origin is a common vertices of the faces of $\partial\Omega_j \setminus \Gamma$. Then $\varphi(x) = |x|^2/2$ directly satisfies (9) and (10), while (11) is trivially satisfied.
For instance we can take $\Omega = \tilde{\Omega} \times]-1,1[$, with $\Omega_i = \tilde{\Omega}_i \times]-1,1[$, $i = 1, \cdots, 4$, where $\tilde{\Omega}$ is the two-dimensional domain.
Rk For these two above examples, particular partitions can be given for which in addition to the above assumptions, the density results hold for adequate choice of ε_j and μ_j.

Fourth example: An almost star-shaped domain

Take the 2D domain $\tilde{\Omega}$ with vertices $A = (-1-\delta/2, 1), B = (1-\delta/2, 1), C = (1-\delta/2, -\delta), D = (1+\delta/2, -\delta), E = -A, F = -B, G = -C, H = -D$ with $\delta > 0$ such that $\delta < 2$. Take the interface I on the line $x_1 = -1$ and define $\tilde{\Omega}_1 = \tilde{\Omega} \cap \{x : x_1 < -1\}$ and $\tilde{\Omega}_2 = \tilde{\Omega} \cap \{x : x_1 > -1\}$. Extend this domain into a prism $\Omega = \tilde{\Omega} \times]-1, 1[$ divided by the two subdomains $\Omega_j = \tilde{\Omega}_j \times]-1, 1[$, we check that Ω is (ε, μ)-substarlike provided that δ is small enough and if

$$\varepsilon_1 \leq \varepsilon_2 \text{ and } \mu_1 \leq \mu_2.$$

Note further that for this example the density of $PH^1(\Omega, \mathcal{P})^3 \cap W_\varepsilon$ (resp. $PH^1(\Omega, \mathcal{P})^3 \cap W_\mu$) into W_ε (resp. into W_μ) always holds.

Outline of the talk

The EC pb

For all $(E_0, H_0) \in \mathcal{H}$, we are looking for a time $T > 0$ and a control $J \in L^2(\Gamma\times]0, T[)^3$ such that the solution (E, H) of

$$\left\{\begin{array}{l} \varepsilon\frac{\partial E}{\partial t} - \mathbf{curl}\, H = 0 \text{ in } Q_T := \Omega\times]0, T[, \\ \mu\frac{\partial H}{\partial t} + \mathbf{curl}\, E = 0 \text{ in } Q_T, \\ \mathrm{div}(\varepsilon E) = \mathrm{div}(\mu H) = 0 \text{ in } Q_T, \\ H \times \nu = J \text{ on } \Sigma_T := \Gamma\times]0, T[, \\ E(0) = E_0,\ H(0) = H_0 \text{ in } \Omega, \end{array}\right. \tag{14}$$

satisfies

$$E(T) = H(T) = 0. \tag{15}$$

The main result

Theorem

If the stability estimate holds, then for $T > 0$ sufficiently large, for all $(E_0, H_0) \in \mathcal{H}$ there exist a control $J \in L^2(\Sigma_T)^3$ satisfying

$$J \cdot \nu = 0 \text{ on } \Sigma_T, \tag{16}$$

such that the solution $(E, H) \in C([0, T], \mathcal{H})$ of (14) is at rest at time T, i.e., satisfies (15).

Its proof is based on Russell's principle.

Pf

For simplicity we prefer to solve the inverse problem: Given $(P_0, Q_0) \in \mathcal{H}$, we are looking for $K \in L^2(\Sigma_T)^3$ satisfying (16) such that the solution $(P, Q) \in C([0, T], \mathcal{H})$ of

$$\left\{\begin{array}{l} \varepsilon \frac{\partial P}{\partial t} - \textbf{curl}\, Q = 0 \text{ in } Q_T, \\ \mu \frac{\partial Q}{\partial t} + \textbf{curl}\, P = 0 \text{ in } Q_T, \\ \mathrm{div}(\varepsilon P) = \mathrm{div}(\mu Q) = 0 \text{ in } Q_T, \\ Q \times \nu = K \text{ on } \Sigma_T, \\ P(T) = P_0,\ Q(T) = Q_0 \text{ in } \Omega, \end{array}\right. \tag{17}$$

satisfies

$$P(0) = Q(0) = 0. \tag{18}$$

Indeed if the above problem has a solution the conclusion follows by setting

$$E(t) = -P(T - t), H(t) = Q(T - t).$$

We solve this problem, using a backward and an inward system with Silver-Müller bc: First given (F_0, I_0) in $\mathcal{H}$, we consider $(F, I) \in C([0, T], \mathcal{H})$ the unique solution of

$$\left\{ \begin{array}{l} \varepsilon \frac{\partial F}{\partial t} - \textbf{curl}\, I = 0 \text{ in } Q_T, \\ \mu \frac{\partial I}{\partial t} + \textbf{curl}\, F = 0 \text{ in } Q_T, \\ \operatorname{div}(\varepsilon F) = \operatorname{div}(\mu I) = 0 \text{ in } Q_T, \\ I \times \nu - (F \times \nu) \times \nu = 0 \text{ on } \Sigma_T, \\ F(T) = F_0,\ I(T) = I_0 \text{ in } \Omega. \end{array} \right. \tag{19}$$

Its existence follows from Corollary 3 by setting $\tilde{E}(t) = -F(T - t)$ and $\tilde{H}(t) = I(T - t)$. Moreover applying Theorem 7 to $(\tilde{E}(t), \tilde{H}(t))$ we get

$$\mathcal{E}(F(t), I(t)) \leq M e^{-\omega(T-t)} \mathcal{E}(F_0, I_0). \tag{20}$$

Second we consider $(G, J) \in C([0, T], \mathcal{H})$ the unique solution of (whose existence and uniqueness still follow from Corollary 3)

$$\left\{ \begin{array}{l} \varepsilon \frac{\partial G}{\partial t} - \textbf{curl}\, J = 0 \text{ in } Q_T, \\ \mu \frac{\partial J}{\partial t} + \textbf{curl}\, G = 0 \text{ in } Q_T, \\ \operatorname{div}(\varepsilon G) = \operatorname{div}(\mu J) = 0 \text{ in } Q_T, \\ J \times \nu + (G \times \nu) \times \nu = 0 \text{ on } \Sigma_T, \\ G(0) = F(0),\ J(0) = I(0) \text{ in } \Omega. \end{array} \right. \tag{21}$$

We now take $P = G - F$ and $Q = J - I$. From (19) and (21), the pair (P, Q) satisfies (17) with

$$K = -(G \times \nu) \times \nu - (F \times \nu) \times \nu. \tag{22}$$

Let us further consider the mapping Λ from $\mathcal{H}$ to $\mathcal{H}$ defined by

$$\Lambda((F_0, I_0)) = (G(T), J(T)).$$

We show that for $T > 0$ such that $d := Me^{-\omega T} < 1$, the mapping $\Lambda - I$ is invertible by proving that $||\Lambda||_{\mathcal{L}(\mathcal{H},\mathcal{H})} < 1$.
Indeed using successively the definition of Λ, the decay of the energy, the initial conditions of problem (21) and the estimate (20) we have

$$\begin{aligned} ||\Lambda((F_0, I_0))||^2_{\mathcal{H}} &= 2\mathcal{E}((G(T), J(T))) \le 2\mathcal{E}((G(0), J(0))) \\ &\le 2\mathcal{E}((F(0), I(0))) \le 2Me^{-\omega T}\mathcal{E}(F_0, I_0) = d||(F_0, I_0)||^2_{\mathcal{H}}. \end{aligned}$$

Consequently $||\Lambda||_{\mathcal{L}(\mathcal{H},\mathcal{H})} \le \sqrt{d} < 1$.
Since $\Lambda - I$ is invertible for any $(P_0, Q_0) \in \mathcal{H}$, there exists a unique $(F_0, I_0) \in \mathcal{H}$ such that

$$(P_0, Q_0) = (P(T), Q(T)) = (G(T), J(T)) - (F(T), I(T)) = (\Lambda - I)(F_0, I_0).$$